Theorem X.13

Let A be a Banach algebra and $a \in A$. Then

(a) $r(a) = \inf_{n \in \mathbb{N}} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}$

(b) A unital $\Rightarrow R(\lambda, a) = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^n}$ for $|\lambda| > r(a)$,

where the series converges absolutely

Proof • Also in (a) WLOG A is unital. So, suppose A is unital.

• $\lambda \in \sigma(a) \overset{L12}{\Rightarrow} \forall n \in \mathbb{N}\ \lambda^n \in \sigma(a^n)$, hence
$|\lambda^n| \le r(a^n) \le \|a^n\|$
$\uparrow$ Prop 8(v)

So, $|\lambda| \le \|a^n\|^{1/n}$

Thus, $r(a) \le \inf_{n \in \mathbb{N}} \|a^n\|^{1/n}$.

• Fix $\varphi \in A^*$. By Prop. 8(v) we have

$$\varphi(R(\lambda, a)) = \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{\lambda^{n+1}} \quad \text{for } |\lambda| > \|a\|$$

Moreover, $\lambda \mapsto \varphi(R(\lambda, a))$ is holomorphic on $\{\lambda;\ |\lambda| > r(a)\}$ by Prop. 8(iv)

Thus, by uniqueness of Laurent series,

$$\varphi(R(\lambda, a)) = \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{\lambda^{n+1}} \quad \text{for } |\lambda| > r(a) \qquad (*)$$

Therefore, $\forall \lambda;\ |\lambda| > r(a)$: $\frac{\varphi(a^n)}{\lambda^{n+1}} \to 0$

hence $\left(\frac{\varphi(a^n)}{\lambda^{n+1}}\right)_{n=1}^{\infty}$ is bdd

$\varphi \in A^*$ arbitrary $\Rightarrow$ by uniform bddness principle (Prop II.27):

$\forall \lambda, |\lambda| > r(a)$: $\left(\frac{a^n}{\lambda^{n+1}}\right)_{n=0}^{\infty}$ is bdd

Fix λ, $|\lambda| > r(a)$. Then there is $C > 0$ s.t.

$\left\| \frac{a^n}{\lambda^{n+1}} \right\| \leq C$ for each $n \in \mathbb{N}$

Hence $\|a^n\| \leq C \cdot |\lambda|^{n+1}$
$\|a^n\|^{1/n} \leq C^{1/n} |\lambda| \cdot |\lambda|^{1/n}$

So, $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq \limsup_{n\to\infty} C^{1/n} |\lambda| \, |\lambda|^{1/n} = |\lambda|$

So, $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a)$
[$|\lambda| > r(a)$ was arbitrary]

Thus:
$\inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \leq \liminf_{n\to\infty} \|a^n\|^{1/n} \leq \limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a) \leq \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}$

Hence (a) follows.

Finally, if $|\lambda| > r(a)$, then

$$\limsup \left\| \frac{a^n}{\lambda^{n+1}} \right\|^{1/n} = \limsup \frac{\|a^n\|^{1/n}}{|\lambda| \, |\lambda|^{1/n}} \overset{(a)}{=} \frac{r(a)}{|\lambda|} < 1,$$

hence the series from (b) converges absolutely.

Moreover, its sum is $R(\lambda, a)$ by (*).

Cor 14 A unital B-alg., $a \in A$, $r(a) < 1 \Rightarrow (e-a)^{-1} = \sum_{n=0}^{\infty} a^n$, the series converges absolutely.

Pf: Since $1 > r(a)$, this follows from Thm 13 (b) applied to $\lambda = 1$.