Def: Let $p(\lambda)=\sum_{j=0}^{n} \alpha_j \lambda^j$ be a polynomial with complex coefficients

- A unital B-algebra, $a \in A$ ..., we set $p(a)=\sum_{j=0}^{n} \alpha_j \cdot a^j$ (where $a^0=e$)
- A non-unital B-algebra, $a \in A$, $p(0)=0$ (i.e. $\alpha_0=0$) ... we set $p(a)=\sum_{j=1}^{n} \alpha_j a^j$

Note: $p(a) \in A$

Lemma 11: A unital, p, q polynomials, $a \in A \Rightarrow (pq)(a)=p(a)\,q(a)$

Proof: ① if p is constant, the equality holds $\ulcorner p \equiv c \Rightarrow p(a)=ce$

$(pq)(a)=(cq)(a)\underset{\text{easy}}{=}c\cdot q(a)=ce\cdot q(a)=p(a)q(a)$ $\lrcorner$

② $p(\lambda)=\lambda \Rightarrow$ the equality holds $\ulcorner q(\lambda)=\sum_{j=0}^{n}\alpha_j\lambda^j$ $(pq)(\lambda)=\sum_{j=0}^{n}\alpha_j\lambda^{j+1}$

$(pq)(a)=\sum_{j=0}^{n}\alpha_j\cdot a^{j+1}$, $p(a)q(a)=a\cdot\sum_{j=0}^{n}\alpha_j\cdot a^j=\sum_{j=0}^{n}\alpha_j a^{j+1}$ $\lrcorner$

③ Fix q $\mathcal{F}=\{p \text{ polynomial};\ (pq)(a)=p(a)q(a)\}$ $\Rightarrow$ $\mathcal{F}$ is a linear subspace, contains constants and $p\in\mathcal{F}\Rightarrow \lambda\mapsto\lambda p(\lambda)\in\mathcal{F}$ $\ulcorner r(\lambda)=\lambda p(\lambda), \lambda\in\mathbb{C}$ $\Rightarrow (rq)(a)\overset{②}{=}a\cdot(pq)(a)\underset{p\in\mathcal{F}}{=}a\cdot p(a)\cdot q(a)\overset{②}{=}r(a)q(a)$ $\lrcorner$

So, $\mathcal{F}$ contains all polynomials

Lemma 12 (on spectrum and polynomials)

A unital Banach algebra, $p(\lambda)=\sum_{j=0}^{n}\alpha_j\cdot\lambda^j$ a polynomial

(a) $p(a)\in G(A) \Leftrightarrow$ the roots of $p \subset \rho(a)$

- p constant $\Rightarrow$
  - $p\neq 0 \Rightarrow p$ has no roots & $p(a)=\alpha_0 e\in G(A)$
  - $p=0 \Rightarrow p(a)=0\notin G(A)$, roots of $p=\mathbb{C}\not\subset\rho(a)$ by Thm 9

- $\deg p\geq 1$ ($\deg p=n$)

$\Rightarrow p(\lambda)=\alpha_n(\lambda-\xi_1)\cdots(\lambda-\xi_n)$, where $\xi_1,\dots,\xi_n$ are roots

Then $p(a)=\alpha_n(a-\xi_1 e)\cdots(a-\xi_n e)$ (by Lemma 11)

Since $a-\xi_1 e,\dots,a-\xi_n e$ commute, Prop. 5(c)

yields $p(a)\in G(A)\Leftrightarrow\{a-\xi_1 e,\dots,a-\xi_n e\}\subset G(A)$

$\Updownarrow$

$\{\xi_1,\dots,\xi_n\}\subset\rho(a)$

(b) $\sigma(p(a)) = p(\sigma(a))$

$\ulcorner$. $\mu \in \sigma(p(a)) \iff \mu e - p(a) \notin G(A) \iff$

(Observe that $\mu e - p(a) = (\mu - p)(a)$, so)

$\iff (\mu - p)(a) \notin G(a) \overset{(a)}{\iff} \exists\, \lambda$, a root of $\mu - p$ s.t. $\lambda \notin \rho(a)$

$\iff \exists\, \lambda \in \sigma(a) : \mu - p(\lambda) = 0 \iff \mu \in p(\sigma(a))$ $\lrcorner$