**Theorem** Let $A$ be a Banach algebra. Then $\forall x \in A$: $\sigma(x)$ is a nonempty compact subset of $\mathbb{C}$

**Proof:** ① $A$ not unital $\Rightarrow$ $\sigma_A(x) = \sigma_{A^+}(x, 0)$, $A^+$ unital. So, WLOG $A$ is unital.

② $A$ unital $\Rightarrow$ $\sigma(x) = \mathbb{C} \setminus \rho(x)$ $\Rightarrow$ it is closed by Prop. 8(ii)

and by Prop. 8(v) $\sigma(x) \subset \overline{U(0, \|x\|)}$

$\Rightarrow \sigma(x)$ is bdd, hence compact

③ $\sigma(x) \neq \emptyset$ ($A$ unital)

Suppose $\sigma(x) = \emptyset$, i.e. $\rho(x) = \mathbb{C}$

Then $\forall \varphi \in A^*: \lambda \mapsto \varphi(R(\lambda, a))$ is an entire function (holomorphic on $\mathbb{C}$)

Moreover, for $|\lambda| > \|a\|$ we have (by prop. 8(v))

$$\|R(\lambda, a)\| = \left\| \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}} \right\| \leq \sum_{n=0}^{\infty} \left\| \frac{a^n}{\lambda^{n+1}} \right\| \leq \sum_{n=0}^{\infty} \frac{\|a\|^n}{|\lambda|^{n+1}} =$$

$$= \frac{\frac{1}{|\lambda|}}{1 - \frac{\|a\|}{|\lambda|}} = \frac{1}{|\lambda| - \|a\|} \longrightarrow 0 \text{ for } \lambda \to \infty$$

Hence, given $\varphi \in A^*$: $\lim_{\lambda \to \infty} \varphi(R(\lambda, a)) = 0$

Hence, by a consequence of the Liouville theorem $\varphi(R(\lambda, a)) = 0$ for $\lambda \in \mathbb{C}$

Corollary II.7 or Corollary V.34

Since $\varphi \in A^*$ is arbitrary, a consequence to H-B theorem yields $R(\lambda, a) = 0$ for $\lambda \in \mathbb{C}$.

It is a contradiction, as $(\lambda e - a)^{-1}$ cannot be $0$.

**Remark:** $G \subset \mathbb{C}$ open $\Rightarrow \{a \in A;\ \sigma(a) \subset G\}$ is open in $A$

- WLOG $A$ unital
- WLOG $G \subsetneq \mathbb{C}$ (if $G = \mathbb{C}$, the statement is trivial)
- Let $\sigma(a) \subset G$. Then $r := \operatorname{dist}(\sigma(a), \mathbb{C} \setminus G) > 0$

Let $K := \overline{U(0, r + \|a\|)} \setminus U(\sigma(a), r)$.

Then $K$ is a compact subset of $\varrho(a)$, $K \neq \emptyset$
($K \supset \{\lambda;\ |\lambda| = r + \|a\|\}$ as $U(\sigma(a), r) \subset U(0, \|a\| + r)$)

$M := \max_{\lambda \in K} \|(\lambda e - a)^{-1}\|$ (cts function on a compact set)

Let now $\|b - a\| < \min\{r, \frac{1}{M}\}$

Then:
- $\|b - a\| < r \Rightarrow \|b\| < \|a\| + r \Rightarrow \sigma(b) \subset U(\|a\| + r)$
- $\lambda \in K \Rightarrow \|(\lambda e - b) - (\lambda e - a)\| = \|b - a\| <$

$< \frac{1}{M} \leq \frac{1}{\|(\lambda e - a)^{-1}\|}$. Hence $\lambda e - b$ is invertible (by Thm. 6 (b))

So, $\lambda \in \varrho(b)$

It follows that $\sigma(b) \cap K = \emptyset$

So, $\sigma(b) \subset U(0, r + \|a\|) \setminus K = U(\sigma(a), r) \subset G$

Gelfand-Mazur theorem (Theorem X. 10)

$A$ unital Banach algebra

$A$ is a field (i.e. $G(A) = A \setminus \{0\}$) $\Leftrightarrow$ $A \approx \mathbb{C}$

Proof: $\Leftarrow$ clear

$\Rightarrow$: $x \in A \Rightarrow \sigma(x) \neq \emptyset$ by Thm 9

Fix $\lambda \in \sigma(x)$. Then $\lambda e - x$ is not invertible, so by the assumption $\lambda e - x = 0$, hence $x = \lambda e$

Therefore, $T : \mathbb{C} \to A$ defined by $T(\lambda) = \lambda e$ is onto.

It is clear that $T$ is an isometric isomorphism of Banach algebras.