Remarks on deficiency indices

(1) S, T symmetric operators $\Rightarrow$ $S \subset T \Leftrightarrow C_S \subset C_T$

$\ulcorner$ This is easy ..... use the formula for C_S and the inverse formula from Thm 27 (c) $\lrcorner$

(2) S ... a closed densely defined symmetric operator

deficiency indices : $\operatorname{codim} D(C_S) = \dim D(C_S)^{\perp}$
$\operatorname{Codim} R(C_S) = \dim R(C_S)^{\perp}$

(a) S is self-adjoint $\Leftrightarrow$ both deficiency indices are 0

$\ulcorner S$ self-adjoint $\overset{\text{Thm 30(a)}}{\Leftrightarrow}$ C_S unitary $\Leftrightarrow$ $D(C_S) = R(C_S) = H$ $\lrcorner$

(b) S maximal symmetric $\Leftrightarrow$ at least one of the deficiency indices is 0

$\ulcorner \Rightarrow$: at least one of the indices are $0 \Rightarrow D(C_S) = H$ or $R(C_S) = H$
Assume $T \supset S$ is symmetric. By (1) $C_T \supset C_S$. But C_T is an isometry, so $C_T = C_S$, hence $T = S$

$\Leftarrow$: Assume both indices are > 0. Take $x \in D(C_S)^{\perp}$, $y \in R(C_S)^{\perp}$
$\|x\| = \|y\| = 1$
$X := \operatorname{span}(D(C_S) \cup \{x\})$, $Y := \operatorname{Span}(R(C_S) \cup \{y\})$
Define $U : X \to Y$ by

$$U(z + \alpha x) = U(z) + \alpha y \quad , \quad z \in D(C_S), \alpha \in \mathbb{C}$$

Then U is an isometry, $D(U) = X$, $R(U) = Y$, $C_S \subsetneq U$

$$S \text{ densely defined} \overset{\text{Thm 27(b)}}{\Rightarrow} R(I - C_S) \text{ is dense}$$

$$\overset{I-C_S \subset I-U}{\Rightarrow} R(I-U) \text{ is dense} \overset{L\,28(b)}{\Rightarrow} I-U \text{ is one-to-one}$$

$$\overset{\text{Thm 29}}{\Rightarrow} \quad U = C_T \text{ for a symmetric } T$$

Finally, $C_S \subsetneq U = C_T$, hence $S \subsetneq T$ (by (1))

So, S is not maximal $\lrcorner$

(c) S admits a self-adjoint extension

$\Leftrightarrow$ the deficiency indices are equal

(i.e., there is a linear isometry $V : D(C_S)^\perp \xrightarrow{\text{onto}} R(C_S)^\perp$)

$\ulcorner \Rightarrow$: $T \supset S$, T self-adjoint.

Then $C_T \supset C_S$, C_T unitary.

Hence $C_T(D(C_S)^\perp) = R(C_S)^\perp$

, because $C_T(D(C_S)) = R(C_S)$

as C_T is an isometry with $D(C_T) = R(C_T) = H$

$\Leftarrow$: Let $V : D(C_S)^\perp \xrightarrow{\text{onto}} R(C_S)^\perp$ be a linear isometry

Then $U(x+y) = C_S(x) + Vy$, $x \in D(C_S)$, $y \in D(C_S)^\perp$

is a unitary operator, $C_S \subset U$

Moreover, as in (b) we see that $I - U$ is one-to-one,

hence $U = C_T$ for a self-adjoint operator T (Thm 30(b))

Then $S \subset T$ $\lrcorner$

③ What happens if S is closed symmetric, but not necessarily densely defined

Deficiency indices make sense

(a) holds as well, by Thm 30 (a)

(b) $\Leftarrow$ holds as well } the same proof

(c) $\Rightarrow$ holds as well }

Moreover:

- $D(C_S) = H$ or $R(C_S) = H \Rightarrow S$ densely defined

⌈ Assume $D(C_S) = H$. We know that $I - C_S$ is one-to-one (Thm 27 (b))

$$\Rightarrow \{0\} = \ker(I - C_S) \overset{L28(b)}{=} \underbrace{D(C_S)}_{=H} \cap R(I - C_S)^\perp = R(I - C_S)^\perp$$

Hence $R(I - C_S)$ is dense, so S is densely defined (Thm 27 (b))

$\Leftarrow$

Assume $R(C_S) = H$ and S is not densely defined

By Thm 27 (b) $R(I - C_S)$ is not dense, so we may find $x \in R(I - C_S)^\perp$, $\|x\| = 1$

$R(C_S) = H \Rightarrow \exists\, y \in D(C_S) : \quad C_S y = x$

Then $0 = \langle x, (I - C_S) y \rangle = \langle x, y \rangle - \langle x, x \rangle$

so, $\langle x, y \rangle = \langle x, x \rangle = 1$

By the equality in Cauchy–Schwarz inequality we deduce $y = x$

Hence $C_S x = x$, $I - C_S$ is not one-to-one. ↯ ⌋

Question:

- Is $\Rightarrow$ in (S) valid without assuming S is densely defined?
  I.e., is any maximal symmetric operator densely defined?
- Is $\Leftarrow$ in (C) valid without assuming S is densely defined?

This is related to the following questions:

- Assume $U: D(U) \to R(U)$ is an isometry, $D(U)$ closed, $D(U) \neq H \neq R(U)$, $I-U$ one-to-one.
  Is there an isometry V such that $U \subsetneq V$, $I-V$ is one-to-one?
- Assume U is as above and, moreover, there is an isometry $W: D(U)^{\perp} \xrightarrow{\text{onto}} R(U)^{\perp}$.

  Is there a unitary operator V extending U such that $I-V$ is one-to-one?