Lemma XII.28 U operator on H, which is an isometry of $D(U)$ onto $R(U)$

(a) $\forall x, y \in D(U): \langle Ux, Uy\rangle = \langle x, y\rangle$. U is unitary $\Leftrightarrow D(U) = R(U) = H$

$\ulcorner$ the formula: Prop. XI.18 (iii) $\Rightarrow$ (iv) (completeness not needed)

the equivalence: Prop. XI.17 (c) $\lrcorner$

(b) $\ker(I-U) = D(U) \cap R(I-U)^\perp$. In particular: $R(I-U)$ dense $\Rightarrow I-U$ one-to-one

$\ulcorner$ $\subset$: $x \in \ker(I-U) \Rightarrow x \in D(U)$ & $(I-U)x = 0$

$$y \in D(U) \Rightarrow \langle x, (I-U)y\rangle = \langle x, y\rangle - \langle x, Uy\rangle \overset{(a)}{=} \langle Ux, Uy\rangle - \langle x, Uy\rangle =$$

$$= \langle \underbrace{Ux - x}_{=0}, Uy\rangle = 0. \text{ So, } x \in R(I-U)^\perp$$

$\supset$: Assume $x \in D(U)$, $x \perp R(I-U)$ $\Rightarrow$

$$\|(I-U)x\|^2 = \langle x - Ux, x - Ux\rangle = \langle \underbrace{x - Ux}_{\in R(I-U)}, x\rangle - \langle x - Ux, Ux\rangle$$

$$= 0 - \langle x, Ux\rangle + \langle Ux, Ux\rangle \overset{(a)}{=} - \langle x, Ux\rangle + \langle x, x\rangle = \langle x, \underbrace{x - Ux}_{\in R(I-U)}\rangle = 0$$

"In particular part": $R(I-U)$ dense $\Rightarrow R(I-U)^\perp = \{0\} \Rightarrow \ker(I-U) = \{0\}$

$\Rightarrow I-U$ one-to-one. $\lrcorner$

Theorem XII.29 U operator on H, isometry $D(U)$ onto $R(U)$. Assume $I-U$ is one-to-one.

Then $S = i(I+U)(I-U)^{-1}$ is symmetric and $C_S = U$.

Moreover, S is densely defined $\Leftrightarrow R(I-U)$ is dense

Proof: Assume $(I-U)$ is one-to-one. Then $(I-U)^{-1}$ is defined, $D((I-U)^{-1}) = R(I-U)$

$R((I-U)^{-1}) = D(I-U) = D(U)$

$D(I+U) = D(U)$

$\Rightarrow S = i(I+U)(I-U)^{-1}$ is well defined,

$D(S) = R(I-U)$, $R(S) = R(I+U)$.

<u>S is symmetric:</u>

- $x,y \in D(U) \Rightarrow \langle (I+U)x, (I-U)y \rangle = \langle x,y \rangle + \langle Ux,y \rangle - \langle x,Uy \rangle - \langle Ux,Uy \rangle$

$$\overset{L28(a)}{=} \langle Ux,Uy \rangle + \langle Ux,y \rangle - \langle x,Uy \rangle - \langle x,y \rangle$$

$$= \langle Ux - x, Uy + y \rangle = -\langle (I-U)x, (I+U)y \rangle$$

- $x,y \in D(S) = R(I-U) \Rightarrow \langle Sx,y \rangle = i \langle (I+U)(I-U)^{-1}x, y \rangle =$

$$= i \langle (I+U)(I-U)^{-1}x, (I-U)(I-U)^{-1}y \rangle$$

$$= -i \langle (I-U)(I-U)^{-1}x, (I+U)(I-U)^{-1}y \rangle =$$

$$= \langle x, i(I+U)(I-U)^{-1}y \rangle = \langle x, Sy \rangle$$

$C_S = U$: $\quad S + iI = i(I+U)(I-U)^{-1} + i\underbrace{(I-U)(I-U)^{-1}}_{I|_{R(I-S)}} =$

$$\overset{P9(iii)}{=} i(I+U+I-U)(I-U)^{-1} = 2i(I-U)^{-1}$$

$$\Rightarrow (S+iI)^{-1} = -\tfrac{1}{2}i(I-U)$$

$$S - iI = i(I+U)(I-U)^{-1} - i\underbrace{(I-U)(I-U)^{-1}}_{I|_{R(I-U)}} \overset{P9(iii)}{=} i(I+U-(I-U))(I-U)^{-1}$$

$$= 2iU(I-U)^{-1}$$

$$\Rightarrow C_S = (S-iI)(S+iI)^{-1} = 2iU(I-U)^{-1} \cdot \left(-\tfrac{1}{2}i\right)(I-U)$$

$$= U(I-U)^{-1}(I-U) = U \cdot I|_{D(U)} = U.$$

<u>Moreover part:</u> $D(S) = R(I-U)$, see the beginning of the proof.

THEOREM 30 (a) S symmetric. Then S is self-adjoint $\Leftrightarrow C_S$ is unitary

(b) U unitary, $I-U$ one-to-one. Then $S = i(I+U)(I-U)^{-1}$ is unitary and $C_S = U$

Proof • (a) $\Rightarrow$: S self-adjoint $\overset{\text{Thm 25}}{\Longrightarrow}$ $\sigma(S) \subset \mathbb{R}$, in particular $\pm i \in \rho(S)$. Thus $S+iI$, $S-iI$ are onto, hence $\underbrace{D(C_S) = R(C_S) = H}_{\text{by Thm 27(a)}}$ $\overset{\text{L28(a)}}{\Longrightarrow}$ C_S unitary

• (b): Assume U is unitary & $I-U$ is one-to-one.

Then $R(I-U)$ is dense: $\{0\} = \operatorname{Ker}(I-U) \overset{\text{L28(b)}}{=} \overbrace{D(U)}^{H} \cap R(I-U)^{\perp} = R(I-U)^{\perp}$

so, $R(I-U)$ is dense

Thm 29 $\Rightarrow$ S symmetric, $C_S = U$, S densely defined

Further, Thm 27(a) $\Rightarrow$ $S+iI$, $S-iI$ are onto $\Rightarrow \pm i \in \rho(S)$

$\overset{\text{Cor 26}}{\Longrightarrow}$ S self-adjoint

• (a) $\Leftarrow$: This follows from (b) & Thm 27(c).