Cayley transform

S symmetric operator on H (not necessarily densely defined, not necessarily closed)

The Cayley transform of S is $C_S := (S - iI)(S+iI)^{-1}$

Theorem XII.27

(a) C_S is an isometry of $D(C_S) = R(S+iI)$ onto $R(C_S) = R(S-iI)$

Γ • $S+iI$ is one-to-one by Lemma XII.24, so $(S+iI)^{-1}$ is defined.

$D((S+iI)^{-1}) = R(S+iI)$
$R((S+iI)^{-1}) = D(S+iI) = D(S)$ $\Big\} \Rightarrow (S+iI)^{-1}$ maps $R(S+iI)$ onto $D(S)$

• $D(S-iI) = D(S)$

$\Rightarrow C_S = (S-iI)(S+iI)^{-1}$ is well defined, $D(C_S) = R(S+iI)$, $R(C_S) = R(S-iI)$

• $\lambda \in \mathbb{C}$, $x \in D(S)$ $\Rightarrow \|(S+\lambda I)x\|^2 = \langle (S+\lambda I)x, (S+\lambda I)x\rangle = \langle Sx, Sx\rangle + \langle Sx, \lambda x\rangle + \langle \lambda x, Sx\rangle + \langle \lambda x, \lambda x\rangle = \|Sx\|^2 + \overline{\lambda}\underbrace{\langle Sx, x\rangle}_{=} + \lambda \underbrace{\langle x, Sx\rangle}_{S \text{ symmetric}} + |\lambda|^2\|x\|^2$

$= \|Sx\|^2 + \overbrace{2\operatorname{Re}\lambda}^{\overline{\lambda}+\lambda} \cdot \langle Sx, x\rangle + |\lambda|^2\|x\|^2$

Apply for $\lambda = \pm i$: $\|(S \pm iI)x\|^2 = \|Sx\|^2 + \|x\|^2$

$\Rightarrow \|(S+iI)x\| = \|(S-iI)x\|$

$\Rightarrow C_S$ is an isometry:

Γ $\|C_S x\| = \|(S-iI)(S+iI)^{-1}x\| = \|(S+iI)(S+iI)^{-1}x\| = \|x\|$

for $x \in D(C_S) = R(S+iI)$ $\lrcorner$

$\lrcorner$

(b) $I - C_S = 2i(S+iI)^{-1}$. In particular, $I - C_S$ is one-to-one and $R(I - C_S) = D(S)$

⌈ $$I - C_S = I - (S-iI)(S+iI)^{-1} = \underbrace{(S+iI)(S+iI)^{-1}}_{I\restriction_{R(S+iI)}} - \underbrace{(S-iI)(S+iI)^{-1}}_{\text{defined on } R(S+iI)}$$

$$\overset{\text{p. 9 (iii)}}{=} (\underbrace{(S+iI)-(S-iI)}_{2iI})(S+iI)^{-1} = 2i(S+iI)^{-1}$$

This proves the formula. It follows that $I-C_S$ is one-to-one (as $(S+iI)^{-1}$ is one-to-one)

and $R(I-C_S) = D(2i(S+iI)^{-1}) = D((S+iI)^{-1}) = R(S+iI)$. ⌋

(c) $S = i(I+C_S)(I-C_S)^{-1}$

⌈ (b) $\Rightarrow$ $I-C_S = 2i(S+iI)^{-1}$, so $(I-C_S)^{-1} = -\frac{1}{2}i(S+iI)$

$I + C_S = 2S(S+iI)^{-1}$ (the same proof as in (b): $(S+iI)+(S-iI)=2S$

$$\Rightarrow (I+C_S)(I-C_S)^{-1} = 2S(S+iI)^{-1}\cdot\left(-\tfrac12 i\right)(S+iI) = -iS\underbrace{(S+iI)^{-1}(S+iI)}_{I\restriction_{D(S+iI)} = I\restriction_{D(S)}}$$

$= -iS\cdot I|_{D(S)} = -iS$. So, $S = i(I+C_S)(I-C_S)^{-1}$ ⌋

(d) C_S is closed $\Leftrightarrow$ S is closed $\Leftrightarrow$ $D(C_S)$ is closed $\Leftrightarrow$ $R(C_S)$ is closed

⌈ Recall that $D(C_S) = R(S+iI)$ and $R(C_S) = R(S-iI)$.

So, by Lemma †11.24: S is closed $\Leftrightarrow$ $D(C_S)$ is closed $\Leftrightarrow$ $R(C_S)$ is closed.

Further C_S is cts (being isometry), so C_S is closed $\Leftrightarrow$ $D(C_S)$ is closed

⌈ $\Rightarrow$: C_S closed, $x\in \overline{D(C_S)}$ $\Rightarrow$ $\exists\,(x_n)\subset D(C_S)$ $x_n\to x$

C_S cts $\Rightarrow$ $(C_S x_n)$ conv $\Rightarrow$ $C_S x_n \to y \in H$

$\overset{C_S \text{ closed}}{\Longrightarrow}$ $x\in D(C_S)$, $y = C_S x$

| $\Leftarrow$ Assume $D(C_S)$ is closed

$(x_n)\subset D(C_S)$, $x_n\to x$ in H, $C_S x_n\to y$ in H

$\overset{D(C_S)\text{ closed}}{\Longrightarrow} x\in D(C_S) \overset{C_S \text{ cts}}{\Longrightarrow} y = C_S x$ ⌋⌋