Lemma XII.24 S symmetric (not necessarily densely defined), $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$\Rightarrow \lambda I - S$ is one-to-one and $(\lambda I - S)^{-1}$ is cts on $R(\lambda I - S)$

Moreover S closed $\Leftrightarrow R(\lambda I - S)$ closed

Proof: (1) Assume $(\lambda I - S)x = 0$. ($\&\ x \in D(S)$). Then $Sx = \lambda x$

Thus

$$\langle Sx, x\rangle = \langle x, Sx\rangle = \langle x, \lambda x\rangle = \bar{\lambda}\langle x, x\rangle$$

$$\overset{\shortparallel}{\langle \lambda x, x\rangle} = \lambda\langle x, x\rangle$$

$\Rightarrow (\lambda - \bar{\lambda})\langle x, x\rangle = 0$. Since $\lambda \notin \mathbb{R}$, we have $\lambda - \bar{\lambda} \neq 0$, so $\langle x, x\rangle = 0$, hence $x = 0$

Thus $\lambda I - S$ is one-to-one

(2) Assume $(\lambda I - S)^{-1}$ is not continuous. Then $\exists\ (x_n) \subset S_H \cap D(S)$

$(\lambda I - S)x_n \to 0$

$\exists\ (y_n) \subset R(\lambda I - S)\ \exists \varepsilon > 0 : y_n \to 0,\ \|(\lambda I - S)^{-1} y_n\| \geq \varepsilon$

- WLOG $\varepsilon = 1$ [replace y_n by $\frac{y_n}{\varepsilon}$]
- WLOG $\|(\lambda I - S)^{-1} y_n\| = 1$ [replace y_n by $\frac{y_n}{\|y_n\|}$]
- take $x_n = (\lambda I - S)^{-1} y_n$

Then also $\langle (\lambda I - S)x_n, x_n\rangle \to 0$

$$\overset{\shortparallel}{\lambda\langle x_n, x_n\rangle - \langle Sx_n, x_n\rangle} = \lambda - \langle Sx_n, x_n\rangle$$

So, $\langle Sx_n, x_n\rangle \to \lambda$. But $\langle Sx_n, x_n\rangle = \langle x_n, Sx_n\rangle = \overline{\langle Sx_n, x_n\rangle}$,

so $\langle Sx_n, x_n\rangle \in \mathbb{R}$, hence $\lambda \in \mathbb{R}$ ↯

(3) $\Leftarrow$: Assume $R(\lambda I - S)$ closed.

$$\left.\begin{matrix} (x_n) \subset D(S) & x_n \to x \\ & Sx_n \to y \end{matrix}\right\} \Rightarrow \underbrace{\lambda x_n - Sx_n}_{=(\lambda I - S)x_n} \to \lambda x - y$$

$\Rightarrow \lambda x - y \in R(\lambda I - S)$

Moreover $(\lambda I - S)^{-1}(\lambda x - y) \overset{(\lambda I - S)^{-1}\text{ cts}}{=} \lim_n (\lambda I - S)^{-1}(\lambda x_n - Sx_n) = \lim_n x_n = x$

So, $\lambda x - y = (\lambda I - S)x = \lambda x - Sx$

Hence $x \in D(S)$ & $Sx = y$. This proves S is closed.

$\Rightarrow$: Assume S is closed. Let $(y_n) \subset R(\lambda I - S)$, $y_n \to y \in H$

$(\lambda I - S)^{-1}$ cts on $R(\lambda I - S)$ $\Rightarrow$ $(\lambda I - S)^{-1} y_n$ is Cauchy in H

$\Rightarrow \exists\, x \in H$: $(\lambda I - S)^{-1} y_n \to x$

$x_n := (\lambda I - S)^{-1} y_n$. Then $(x_n) \subset D(S)$, $x_n \to x$)

$$\left.\begin{array}{l}(\lambda I - S) x_n = y_n \to y \\ \underbrace{\lambda x_n}_{\downarrow \lambda x} - S x_n\end{array}\right\} \Rightarrow \quad S x_n \to \lambda x - y\ )$$

$\hookrightarrow$ S closed $\Rightarrow$ $x \in D(S)$

$S x = \lambda x - y$

$\Rightarrow y = \lambda x - S x = (\lambda I - S) x \in R(\lambda I - S)$

<u>Theorem XII.25</u> T self-adjoint $\Rightarrow$ $\emptyset \neq \sigma(T) \subset \mathbb{R}$

<u>Proof</u> • $\lambda \in \mathbb{C} \Rightarrow (\lambda I - T)^* = \bar{\lambda} I - T^* = \bar{\lambda} I - T$ (P17(b))

• $\lambda \in \mathbb{C} \setminus \mathbb{R} \Rightarrow \{0\} = \underset{\substack{\uparrow \\ L24}}{\mathrm{Ker}(\bar{\lambda} I - T)} \underset{\substack{\uparrow \\ P18\,f}}{=} R(\lambda I - T)^{\perp}$

$\Rightarrow R(\lambda I - T)$ is dense

So: $\lambda I - T$ closed (as T is closed, being self-adjoint; by P11(a))

$\lambda I - T$ one-to-one, $(\lambda I - T)^{-1}$ cts on $R(\lambda I - T)$ (by L24)

$R(\lambda I - T)$ dense

$\underset{P13}{\Rightarrow}$ $R(\lambda I - T) = H$

$(\lambda I - T)^{-1} \in \mathcal{L}(H)$

$\Rightarrow \lambda \in \varrho(T)$.

Hence $\sigma(T) \subset \mathbb{R}$.

• Assume $\sigma(T) = \emptyset$. Then $T^{-1} \in \mathcal{L}(H)$ & $\sigma(T^{-1}) = \{0\}$ (by L15)

But T^{-1} is self-adjoint (P23 (d) or (e))

So, $\|T^{-1}\| \underset{\substack{\uparrow \\ \text{Prop XI.3}}}{=} r(T^{-1}) = 0$ ↯

Corollary XII.26 T densely defined on H. TFAE

(i) T selfadjoint

(ii) T symmetric, $\sigma(T) \subset \mathbb{R}$

(iii) T symmetric, $\exists \lambda \in \mathbb{C} \setminus \mathbb{R}$: $\lambda, \bar{\lambda} \in \rho(T)$.

Pf: (i) $\Rightarrow$ (ii) follows from Thm 25

(ii) $\Rightarrow$ (iii) trivial

(iii) $\Rightarrow$ (i) Assume $\lambda \in \mathbb{C} \setminus \mathbb{R}$ s.t. $\lambda, \bar{\lambda} \in \rho(T)$

$\Rightarrow$ $(\lambda I - T)$, $(\bar{\lambda} I - T)$ are one-to-one and onto.

Moreover, $(\lambda I - T)^* \underset{P17(b)}{=} \bar{\lambda} I - T^* \underset{T \subset T^*}{\supset} \bar{\lambda} I - T$

$(\bar{\lambda} I - T)^* = \lambda I - T^* \supset \lambda I - T$

Moreover $\ker (\lambda I - T)^* \underset{P18}{=} R(\lambda I - T)^\perp \underset{\lambda I - T \text{ is onto}}{=} \{0\} \Rightarrow \bar{\lambda} I - T^*$ is one-to-one

$\underset{\text{one-to-one}}{\bar{\lambda} I - T^*} \supset \underset{\text{onto}}{\bar{\lambda} I - T} \Rightarrow \bar{\lambda} I - T^* = \bar{\lambda} I - T$

$\Rightarrow T^* = T$

$\Rightarrow$ T self-adjoint.