Lemma XII.22 T self-adjoint $\Rightarrow$ T maximal symmetric

Proof T self-adjoint $\Rightarrow$ $T^* = T$, in part. T is densely defined

Assume S is symmetric, $T \subset S$

Then S is densely defined, so $S \subset S^*$

On the other hand $S^* \subset T^*$ (P17(a))

So : $T \subset S \subset S^* \subset T^* = T \Rightarrow S = T$

Proposition XII.23 T symmetric, densely defined. Then:

(a) $\bar{T}$ is symmetric

$\ulcorner$ T densely defined $\Rightarrow$ $T \subset T^*$, T^* closed (P21(a))

$\Rightarrow$ T has a closed extension and $\bar{T} \subset T^*$

Further, $G(T^*) \overset{L19}{=} V(G(T)^\perp) = V(G(\bar{T})^\perp) = G((\bar{T})^*)$

$A^\perp = (\bar{A})^\perp$

$\Rightarrow (\bar{T})^* = T^*$, so $\bar{T} \subset (\bar{T})^*$, hence $\bar{T}$ is symmetric $\lrcorner$

(b) $D(T) = H \Rightarrow T \in L(H)$, T self-adjoint

$\ulcorner$ $T \subset T^*$, $D(T) = H \Rightarrow T = T^*$, hence T is self-adjoint.

Moreover, since $T = T^*$, T is closed, hence $T \in L(H)$ (P10(a)) $\lrcorner$

(c) $R(T)$ dense $\Rightarrow$ T is one-to-one

$\ulcorner$ $\ker T \underset{T \subset T^*}{\subset} \ker T^* \underset{P18}{=} R(T)^\perp = \{0\}$ if $R(T)$ is dense $\lrcorner$

(d) $R(T) = H \Rightarrow T$ is one-to-one, self-adjoint and $T^{-1} \in \mathcal{L}(H)$

$\ulcorner$ • $R(T) = H \Rightarrow \ker T^* \overset{P18}{=} R(T)^\perp = \{0\} \Rightarrow T^*$ one-to-one

$T \subset T^*$, T onto, T^* one-to-one $\Rightarrow T = T^*$, so T is self-adjoint, one-to-one

• T closed, one-to-one, onto $\Rightarrow T^{-1} \in \mathcal{L}(H)$ by P13

being self-adjoint $\lrcorner$

(e) T self-adjoint, one-to-one $\Rightarrow T^{-1}$ self-adjoint

$\ulcorner$ $T^* = T$, T one-to-one $\Rightarrow \{0\} = \ker T = \ker T^* = R(T)^\perp \Rightarrow R(T)$ dense

$T = T^*$ P18

Thus T^{-1} is densely defined and by L20: $(T^{-1})^* = (T^*)^{-1} = T^{-1}$

$\Rightarrow T^{-1}$ self-adjoint $\lrcorner$