Proposition XII.14 : X a Banach space, T an operator on X ($T: D(T) \to X$, where $D(T) \subset\subset X$)

(a) $\mu \in \rho(T)$, $|\lambda - \mu| < \frac{1}{\|(\mu I - T)^{-1}\|}$ $\Rightarrow$ $\lambda \in \rho(T)$

$$\& \quad (\lambda I - T)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda-\mu)^n \left((\mu I - T)^{-1}\right)^{n+1}$$

$\ulcorner$ $\mu \in \rho(T) \Rightarrow$ $\mu I - T$ maps $D(T)$ bijectively on X and $(\mu I - T)^{-1} \in \mathcal{L}(X)$

Assume $|\lambda - \mu| < \frac{1}{\|(\mu I - T)^{-1}\|}$

Then $\lambda I - T = (\lambda - \mu) I + \mu I - T = (\lambda-\mu) \overbrace{(\mu I - T)^{-1}(\mu I - T)}^{= I|_{D(T)}} + \mu I - T$

Prop XII.9(iii)

$$= \Big(\underbrace{(\lambda - \mu)(\mu I - T)^{-1}}_{\in \mathcal{L}(X),\ \|\cdot\| < 1} + I \Big) (\mu I - T)$$

$\Rightarrow$
- $\mu I - T$ maps $D(T)$ onto X (and is one-to-one)
- $(\lambda - \mu)(\mu I - T)^{-1} + I$ is invertible by Lemma X.6(a) and $\left((\lambda-\mu)(\mu I - T)^{-1} + I\right)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda-\mu)^n \left((\mu I - T)^{-1}\right)^n$

So $\lambda I - T$ maps $D(T)$ onto X in a one-to-one manner and

$$(\lambda I - T)^{-1} = (\mu I - T)^{-1} \sum_{n=0}^{\infty} (-1)^n (\lambda-\mu)^n \left((\mu I - T)^{-1}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (\lambda-\mu)^n \left((\mu I - T)^{-1}\right)^{n+1}$$ $\lrcorner$

(b) $\rho(T)$ is an open subset of $\mathbb{C}$, $\sigma(T)$ is a closed subset of T

$\ulcorner$ by (a): $\mu \in \rho(T) \Rightarrow U\left(\mu, \frac{1}{\|(\mu I - T)^{-1}\|}\right) \subset \rho(T)$; $\sigma(T) = \mathbb{C} \setminus \rho(T)$ $\lrcorner$

(c) $\lambda \mapsto (\lambda I - T)^{-1}$ is cts on $\varrho(T)$

$\ulcorner$ Assume $\mu \in \varrho(T)$, $|\lambda - \mu| < \frac{1}{\|(\mu I - T)^{-1}\|}$.

(a) $\Rightarrow$ $\lambda \in \varrho(T)$ & $\|(\lambda I - T)^{-1} - (\mu I - T)^{-1}\| = \|\sum_{n=1}^{\infty} (-1)^n (\lambda - \mu)^n ((\mu I - T)^{-1})^{n+1}\|$

$$\leq \sum_{n=1}^{\infty} |\lambda - \mu|^n \|(\mu I - T)^{-1}\|^{n+1} = \frac{|\lambda - \mu| \|(\mu I - T)^{-1}\|^2}{1 - |\lambda - \mu| \|(\mu I - T)^{-1}\|} \xrightarrow[\lambda \to \mu]{} 0 \quad \lrcorner$$

(d) $f \in X^*$, $x \in X \Rightarrow \lambda \mapsto f((\lambda I - T)^{-1} x)$ is holomorphic on $\varrho(T)$

$\ulcorner$ Fix $f \in X^*$, $x \in X$. $\mu \in \varrho(T) \overset{(a)}{\Rightarrow}$ for $\lambda \in U\left(\mu, \frac{1}{\|(\mu I - T)^{-1}\|}\right)$ we have

$$f((\lambda I - T)^{-1} x) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \mu)^n \cdot f(((\mu I - T)^{-1})^{n+1} x).$$

It is a power series. $\lrcorner$

<u>Proposition XII.15</u> : T closed operator on X, $\sigma(T) = \emptyset \Rightarrow T^{-1} \in \mathcal{L}(X)$, $\sigma(T^{-1}) = \{0\}$

<u>Proof:</u>

- $\sigma(T) = \emptyset \Rightarrow \varrho(T) = \mathbb{C} \Rightarrow 0 \in \varrho(T) \Rightarrow T^{-1} \in \mathcal{L}(X)$
- $0 \in \sigma(T^{-1})$: $D(T) \subsetneq X$ (otherwise $T \in \mathcal{L}(X)$ by Prop XII.10(a); then $\sigma(T) \neq \emptyset$ by Theorem X.9)

  $\Rightarrow T^{-1}$ is not onto
- $\lambda \in \mathbb{C} \setminus \{0\} \Rightarrow \lambda I - T^{-1} \underset{TT^{-1} = I}{=} \lambda T \cdot T^{-1} - T^{-1} \underset{\text{Prop. XII.9 (iii)}}{=} (\lambda T - I) T^{-1} =$

  $= \lambda (T - \frac{1}{\lambda} I) T^{-1}$

  - $D(T^{-1}) = X$, $R(T^{-1}) = D(T)$, T^{-1} one-to-one
  - $D(T - \frac{1}{\lambda} I) = D(T)$, $T - \frac{1}{\lambda} I$ is one-to-one, $R(T - \frac{1}{\lambda} I) = X$ (because $\frac{1}{\lambda} \notin \sigma(T)$)

So, combining these facts we get that
$\lambda I - T^{-1}$ is one-to-one and maps X onto X,
so it is invertible (by Corollary III.10).
Hence $\lambda \in \varrho(T^{-1})$.