

## Remarks on Proposition XII.13 for non-closed operators:

one-to-one

Assume  $T$  is an operator from  $X$  to  $Y$  which is not closed:

- Condition (i) fails:

If  $R(T) = Y$  and  $T^{-1} \in \mathcal{L}(Y, X)$ , then  $T^{-1}$  is closed and hence  $T$  is also closed.

- Condition (ii) may hold.

In this case (i) and (ii) fail: (i) fails always (see above) and (ii) in this case reduces to (i).

[By (ii)  $R(T) = Y$ . (ii) says that  $R(T)$  is closed and  $T^{-1}$  is continuous on  $R(T)$ . But we have even  $R(T) = Y$ , so  $T^{-1}$  is continuous on  $Y$ , i.e.  $T^{-1} \in \mathcal{L}(Y, X)$ . This implies (i).]

EXAMPLE 1:  $T: X \rightarrow X$  discontinuous linear bijection.

If  $\dim X = \infty$ , it exists ... take an algebraic basis  $(x_i)_{i \in \mathbb{I}}$  of  $X$  and define  $T$  by  $x_i \mapsto \lambda_i x_i$ , where  $(\lambda_i)_{i \in \mathbb{I}}$  is unbounded.

Then  $T$  has no closed extension [  $T$  is not closed and has no proper extension ]

If  $T$  has a closed extension, then  $\overline{T}$  is not one-to-one

[  $x \in D(\overline{T}) \setminus D(T) \Rightarrow \overline{T}x \in Y = R(T) \Rightarrow \exists y \in D(T)$

$Ty = \overline{T}x$ . Hence  $\overline{T}$  is not one-to-one. ]

Example 2:  $X = \ell_2$ ,  $S: \ell_2 \rightarrow \ell_2$   $S(x_1, x_2, \dots) = (x_2, x_3, \dots)$

$\Rightarrow S \in \mathcal{L}(X)$ ,  $S$  is not one-to-one,  $S$  is onto.

Let  $\varphi \in X^\#$  be a discontinuous linear functional.

Set  $D(T) = \{x \in \ell_2; x_1 = \varphi(Sx)\}$

$T = S \upharpoonright D(T)$

Then  $T$  is one-to-one:  $x \in D(T); Tx = 0$

$\Rightarrow x_2 = x_3 = \dots = 0, x_1 = \varphi(Sx) = 0 \Rightarrow x = 0$

T is onto:  $x \in \ell_2$   $y = (\varphi(x), x_1, x_2, \dots)$

$\Rightarrow Sy = x, y \in D(T), \text{ so } Ty = x.$

D(T) is dense: We know that  $\ker \varphi$  is dense

$x \in \ell_2, \epsilon > 0$

.....  $\{z \in \ell_2; \varphi(z) = x\}$  is dense

so, find  $z \in \ell_2, \varphi(z) = x, \|z - Sx\| < \epsilon$   
set  $y = (x_1, z_1, z_2, \dots)$  then  $y \in D(T)$   
 $\|y - x\| = \|z - Sx\| < \epsilon$

Condition (cii) may hold.

In this case (ci) and (cii) fail. (ci) fails always (see above)  
(cii) in this case reduces to (ci)

T may have closed extension. In this case  $\overline{T}$  satisfies (ci) - (cii)

$\Gamma R(T)$  dense,  $T^{-1}$  continuous on  $R(T) \Rightarrow T^{-1}$  may be  
extended to  $S \in L(Y, X)$ . Moreover,  $S = \overline{T^{-1}}$

(by construction of the extension). Hence,  $G(S) = G(T^{-1})$

So, if T has a closed extension, it must be  $S^{-1}$ .

Therefore, T is one-to-one and onto

EXAMPLE 3  $X = \ell_2, D(T) = c_{00} =$  finitely supported vectors

$$Tx = x, x \in D(T)$$

T need not have a closed extension:

EXAMPLE 4:  $X = \ell_2, \varphi$  discontinuous linear functional

$$Tx = (\varphi(x), x_1, x_2, \dots)$$

T one-to-one

$D(T) = X, T$  discontinuous  $\Rightarrow \overline{T}$  not closed, no proper extension

$\overline{R(T)} = X, T^{-1}$  continuous, as T is the inverse of the operator  
from EXAMPLE 2 above.