Theorem X.7

A unital Banach algebra

(1) $G(A)$ is an open subset of A

⌈ By Lemma 6(3) : $x \in G(A) \Rightarrow U(x, \frac{1}{\|x^{-1}\|}) \subset G(A)$ ⌋

(2) $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$ onto $G(A)$

⌈ $\varphi(x) := x^{-1}$. Then $\varphi(\varphi(x)) = x$, so φ is a bijection $G(A)$ onto $G(A)$ and $\varphi^{-1} = \varphi$.
So, it is enough to show that φ is cts:

Suppose $x_n \to x$ in $G(A)$. There is n_0 s.t. for $n \geq n_0$ we have $\|x_n - x\| < \frac{1}{\|x^{-1}\|}$

Then, by Lemma 6(3), we have for $n \geq n_0$ [take $h = x_n - x$]

$$\|x_n^{-1} - x^{-1}\| \leq \frac{\|x^{-1}\|^2 \, \|x_n - x\|}{1 - \|x^{-1}\| \, \|x_n - x\|} \xrightarrow{n\to\infty} 0$$

Thus $x_n^{-1} \to x^{-1}$. ⌋

(3) $(x_n) \subset G(A)$, $x_n \to x \in A \setminus G(A) \Rightarrow \|x_n^{-1}\| \to \infty$

⌈ Since $x_n \in G(A)$, $x \notin G(A)$, by Lemma 6(3)

we have $\|x_n - x\| \geq \frac{1}{\|x_n^{-1}\|}$. Since $\|x_n - x\| \to 0$,

we deduce $\frac{1}{\|x_n^{-1}\|} \to 0$, so $\|x_n^{-1}\| \to \infty$ ⌋