FUNCTIONAL ANALYSIS 2

WINTER SEMESTER 2016/2017

PROBLEMS TO CHAPTER XI

Problems to Section XI.1 – lattice of locally convex topologies

Problem 1. Let X be a vector space.

(1) Let ρ be a nonzero seminorm on X. Show that there is a nonzero $f \in X^\#$ such that $|f| \leq \rho$.
(2) Suppose that $\mathcal{T}$ is a locally convex topology on X with $(X, \mathcal{T})^* = \{0\}$. Show that $\mathcal{T}$ is the indiscrete topology.

`Hint:` *(1) Use the Hahn-Banach extension theorem. (2) Suppose $\mathcal{T}$ is not the indiscrete topology. Show that there is a nonzero $\mathcal{T}$-continuous seminorm.*

Problem 2. Let X be a vector space and $M \subset\subset X^\#$ separating points of X. Show that $(X, \mu(X, M))$ is metrizable if and only if M can be covered by a sequence of absolutely convex $\sigma(M, X)$-compact sets.

`Hint:` *To show the only if part use the Banach-Alaoglu theorem. To show the if part use Theorem XI.6 and Proposition XI.7.*

Problem 3. Let X be a LCS. Let $\beta(X^*, X)$ be the topology (on X^*) of uniform convergence on bounded subsets of X.

(1) Show that the topology $\beta(X^*, X)$ is the same for any admissible topology on X.
(2) Describe the topology $\beta(X^*, X)$ in case X is a normed space.
(3) Show that the topology $\beta(X^*, X)$ may or may not be admissible on (X^*, w^*).
(4) Characterize the normed spaces for which $(X^*, \beta(X^*, X))^* = \varkappa(X)$.

`Hint:` *(1) Use Theorem VI.8.*

Problem 4. Let X be a normed space and $\mathcal{B}$ a nonempty family of bounded subsets of X. Let $\mathcal{T}$ be the topology (on X^*) of uniform convergence on sets from $\mathcal{B}$. Show that

$$(X^*, \mathcal{T})^* = \operatorname{span} \bigcup_{A \in \mathcal{B}} \overline{\operatorname{aco} \varkappa(A)}^{\sigma(X^{**}, X^*)}.$$

`Hint:` *Use Lemma XI.5.*

Problem 5. Let X be a normed space. Equip X^* with the topology $\mathcal{T}$ of uniform convergence on countable bounded subsets of X.

(1) Describe $(X^*, \mathcal{T})^*$ and show that $\overline{\varkappa(X)} \subset (X^*, \mathcal{T})^* \subset X^{**}$.
(2) Suppose X is separable. Show that $\mathcal{T}$ coincide with the norm topology.
(3) Suppose X is nonseparable. Show that $\mathcal{T}$ is strictly weaker than the norm topology.
(4) Let $X = c_0(\Gamma)$, where Γ is an uncountable set. Consider the canonical identification $X^* = \ell^1(\Gamma)$ and $X^{**} = \ell^\infty(\Gamma)$. Describe $(X^*, \mathcal{T})^*$ as a subset of $\ell^\infty(\Gamma)$.

`Hint:` *(1,4) Use Problem 4. (3) Use Lemmata XI.2 and XI.5.*

Problem 6. Let $X = c_0$ or $X = \ell^p$ for some $p \in [1,\infty]$. Assume X is equipped with the topology τ_p of pointwise convergence on $\mathbb{N}$.

(1) Describe $M = (X, \tau_p)^*$.
(2) Compare the topologies $\sigma(X, M)$, τ_p and $\mu(X, M)$.

`Hint:` *(1) Use Theorem VI.4. (2) Use Proposition XI.7.*

Problem 7. Let Γ be an infinite set and let $X = c_{00}(\Gamma)$ be the vector space of functions $f : \Gamma \to \mathbb{F}$ which have nonzero value only at finitely many elements of Γ. Consider X equipped with the topology τ_p of pointwise convergence on Γ.

(1) Let A be a convex τ_p-compact subset of X. Show that $\operatorname{span} A$ has finite dimension.
(2) Find a τ_p-compact subset $A \subset X$ such that $\operatorname{span} A$ has infinite dimension.
(3) Deduce that a closed convex hull of a compact subset of (X, τ_p) need not be compact.

`Hint:` *(1) Proceed by contradiction. Suppose that* $\operatorname{span} A$ *has infinite dimension. Without loss of generality* $0 \in A$*. The first step is then to find sequences* (f_n) *in* A *and* (γ_n) *is* Γ *such that* $f_n(\gamma_n) \neq 0$ *and* $f_k(\gamma_n) = 0$ *for* $n > k$*.*

Problem 8. Let Γ be an infinite (possibly uncountable) set. Let $X = c_{00}(\Gamma)$ be equipped with the topology τ_p of pointwise convergence on Γ.

(1) Describe $M = (X, \tau_p)^*$.
(2) Describe the weak*-topology on M (i.e., the topology $\sigma(M, X)$).
(3) Compare the topologies $\sigma(X, M)$, τ_p and $\mu(X, M)$.

`Hint:` *(1) Use Theorem VI.4. (3) Use Theorem XI.6 and Problem 7.*

Problem 9. Let Γ be an infinite (possibly uncountable) set. Let $X = c_0(\Gamma)$ or $X = \ell^p(\Gamma)$ for some $p \in [1,\infty)$. Assume X is equipped with the topology τ_p of pointwise convergence on Γ.

(1) Describe $M = (X, \tau_p)^*$.
(2) Describe the weak*-topology on M (i.e., the topology $\sigma(M, X)$).
(3) Characterize the bouded sets in $(M, \sigma(M, X))$.
(4) Describe the weak*-topology on bounded subsets on M.
(5) Compare the topologies $\sigma(X, M)$, τ_p and $\mu(X, M)$.

`Hint:` *(1) Use Theorem VI.4. (2) Use the known weak*-topologies of the respective normed spaces. (3) Show that they are bounded in the respective norm. (4) The dual unit ball of* $(X, \|\cdot\|)^*$ *is weak*-compact and the topology of uniform convergence on* Γ *is a weaker Hausdorff topology. (5) Use Theorem XI.6, (4) and Problem 7.*

Problem 10. Let K be a compact Hausdorff space and $X = (\mathcal{C}(K), \tau_p)$.

(1) Show that $X^* = c_{00}(K)$ and interpret this equality.
(2) Show that $A \subset c_{00}(K)$ is weak*-bounded if and only if it is bounded in $\ell^1(K)$.
(3) Let A be a convex weak*-compact subset of $c_{00}(K)$. Show that $\operatorname{span} A$ has finite dimension.
(4) Deduce that $\mu(X, X^*) = \tau_p$.

`Hint:` *(1) Use Theorem VI.4. (2) Note that* $c_{00}(K)$ *is a subspace of* $(\mathcal{C}(K), \|\cdot\|_\infty)^*$*, the weak*-bounded and norm bounded sets in this dual coincide, its weak* topology restricted to* $c_{00}(K)$ *coincides with the weak* topology of* $c_{00}(K) = X^*$ *and the norm restricted to* $c_{00}(K)$ *coincides with the* ℓ^1*-norm. (3) Start similarly as in Problem 7(1) – assume that* $\operatorname{span} A$ *has infinite dimension,*

$0 \in A$ and construct the respective sequences (f_n) in $c_{00}(K)$ and (γ_n) in K. Find a sequence of positive numbers (t_n) such that $\sum t_n \le 1$, $\sum t_n f_n$ converges absolutely in $(\mathcal{C}(K), \|\cdot\|_\infty)^*$ but the sum does not belong to $c_{00}(K)$. Deduce that A cannot be compact. (4) Use (3) and Theorem XI.6.

PROBLEMS TO SECTION XI.2 – bw^*-TOPOLOGY VS. w^*-TOPOLOGY

Problem 11. Let $p \in [1, \infty)$ and let $q \in (1, \infty]$ be the conjugate exponent. Consider the space $X = \ell^p$ and its dual represented as $X^* = \ell^q$. Let (a_n) be a sequence of nonzero numbers. Denote by $\boldsymbol{e}_n^* \in \ell^q$ the canonical unit vectors.

(1) Show that
$$a_n \boldsymbol{e}_n^* \xrightarrow{w^*} 0 \iff (a_n) \text{ is bounded.}$$
(2) Show that
$$0 \in \overline{\{a_n \boldsymbol{e}_n^*; n \in \mathbb{N}\}}^{bw^*} \iff \liminf_n |a_n| < \infty.$$
(3) Show that
$$0 \in \overline{\{a_n \boldsymbol{e}_n^*; n \in \mathbb{N}\}}^{w^*} \iff (\tfrac{1}{a_n}) \notin \ell^p$$
(4) Find a countable weak*-dense subset of $X^* = \ell^q$ which is bw^*-closed.

`Hint:` *(4) Observe that X^* is weak* separable. Let $(\boldsymbol{x}_n)$ be a fixed weak*-dense sequence. Fix a sequence (a_n) of positive numbers such that $a_n \to \infty$ and $(\frac{1}{a_n}) \notin \ell^p$. Take, for example, the set $\{\boldsymbol{x}_n + a_k \boldsymbol{e}_k; k, n \in \mathbb{N} \ \& \ \|\boldsymbol{x}_n + a_k \boldsymbol{e}_k\| > n\}$.*

Problem 12. Let $X = c_0$ and consider X^* represented as ℓ^1. Let (a_n) be a sequence of nonzero numbers. Denote by $\boldsymbol{e}_n^* \in \ell^1$ the canonical unit vectors.

(1) Show that
$$a_n \boldsymbol{e}_n^* \xrightarrow{w^*} 0 \iff (a_n) \text{ is bounded.}$$
(2) Show that
$$0 \in \overline{\{a_n \boldsymbol{e}_n^*; n \in \mathbb{N}\}}^{bw^*} \iff 0 \in \overline{\{a_n \boldsymbol{e}_n^*; n \in \mathbb{N}\}}^{w^*} \iff \liminf_n |a_n| < \infty.$$

Problem 13. Let X be an infinite dimensional normed linear space.

(1) Show that the bw^*-topology on X^* is strictly stronger than the weak* topology.
(2) Suppose X is separable. Show that there is a countable set $C \subset X^*$ such that $0 \in \overline{C}^{w^*} \setminus \overline{C}^{bw^*}$.
(3) Suppose X is separable. Show that there is a countable subset of X^* which is bw^*-closed and weak* dense in X^*.

`Hint:` *(1) Let (x_n) be a linearly independent sequence in S_X. Using Proposition XI.11 show that the set $U = \{f \in X^*; (\forall n \in \mathbb{N})(|f(x_n)| < n)\}$ is bw^*-open but 0 does not belong to the weak*-interior of U. (2) Let U be set from (1). Then $0 \in \overline{X^* \setminus U}^{w^*}$. Using separability of X show that $X^* \setminus U$ has a countable weak* dense subset C. (3) Let (x_n^*) be a weak*-dense sequence in X^* and C be the set from (2). Consider the set $\{x_n^* + y^*; n \in \mathbb{N}, y^* \in C \ \& \ \|\boldsymbol{x}_n^* + y^*\| > n\}$.*

Problem 14. Let X be a Banach space and $Y \subset\subset X^*$ a weak*-dense subspace of finite codimension.

(1) Show that $Y_\perp = \{0\}$ and $Y^\perp \subset X^{**}$ is a finite-dimensional subspace.
(2) Deduce that $C = \operatorname{dist}(S_{Y^\perp}, \varkappa(X)) > 0$.
(3) Deduce that Y is $(1 + \frac{1}{C})$-norming.

Hint: *(2) $S_{Y^\perp}$ is compact and $\varkappa(X)$ is closed. (3) Fix $x \in S_X$. Observe that $\tilde{q}_Y(x) = \|\varkappa(X)|_Y\|$ ($\tilde{q}_Y$ is defined in the final remark of Section XI.2). Use Hahn-Banach theorem to find $F \in X^{**}$ such that $F|_Y = \varkappa(X)|_Y$ and $\|F\| = \tilde{q}_Y(x)$. Then $F - \varkappa(x) \in Y^\perp$ and $\|F - \varkappa(x)\| \geq 1 - \tilde{q}_Y(x)$. Deduce that $C \leq \frac{\tilde{q}_Y(x)}{1-\tilde{q}_Y(x)}$, so $\tilde{q}_Y(x) \geq 1 + \frac{1}{C}$.*

Problem 15. Let X be a Banach space and $F \in X^{**} \setminus \varkappa(X)$. Show that $\ker F$ is a norming subspace of X^*.

Hint: *Use Problem 14.*

Problem 16. Let X be a non-complete normed space. Show that there is a weak*-dense subspace of X^* of codimension one which is not norming.

Hint: *Take $\ker F$ for $F \in \overline{\varkappa(X)} \setminus \varkappa(X)$.*

Problem 17. Consider the space ℓ^1 as the dual to c_0, using the canonical representation. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoint infinite subsets of $\mathbb{N}$. For each $n \in \mathbb{N}$ fix some $k_n \in A_n$. Show that the subspace

$$Y = \{\boldsymbol{x} \in \ell^1; \forall n \in \mathbb{N} : n x_{k_n} + \sum_{j \in A_n \setminus \{k_n\}} x_j = 0\}$$

is weak* dense but not norming.

Hint: *Show that the point $\boldsymbol{x}$ defined by $x_k = \begin{cases} \frac{1}{n^2} & k = k_n \\ 0 & \text{otherwise} \end{cases}$ does not belong to $\overline{Y \cap rB_{\ell^1}}^{w^*}$ for any $r > 0$.*

Problem 18. Let X be a separable Banach space and $A \subset X^*$. Show that the set

$$\bigcup_{r>0} \overline{A \cap rB_{X^*}}^{w^*}$$

consist exactly of limits of weak*-convergent sequences from A.

Hint: *For one inclusion use the uniform boundedness principle, for the converse use metrizability of (B_{X^*}, w^*).*

Problems to Section XI.3 – compact convex set, extreme points

Problem 19. Let X be a Banach space and $K \subset X$ a nonempty weakly compact convex set. Show that $\overline{\operatorname{co}\operatorname{ext} K}^{\|\cdot\|} = K$.

Hint: *Combine Krein-Milman and Mazur theorems.*

Problem 20. Let X be a reflexive Banach space. Show that $\overline{\operatorname{co}\operatorname{ext} B_X}^{\|\cdot\|} = B_X$.

Hint: *Use Problem 19.*

Problem 21. Let $p \in (1, \infty)$ and let μ be any σ-additive measure such that there exists a measurable set A with $0 < \mu(A) < \infty$. Show the set $\operatorname{ext} B_{L^p(\mu)}$ coincides with the unit sphere.

Problem 22. Let μ be a σ-additive measure which is not constant zero.

(1) Describe the extreme points of $B_{L^\infty(\mu)}$ both in the real and complex cases.

(2) Is $\overline{\operatorname{co}\operatorname{ext} B_{L^\infty(\mu)}}^{\|\cdot\|} = B_{L^\infty(\mu)}$?

`Hint:` *(2) Show that simple functions are dense in $L^\infty(\mu)$ and use this fact.*

Problem 23. Let Γ be a set containing at least two points.

(1) Describe $\operatorname{ext} B_{\ell^1(\Gamma,\mathbb{R})}$.
(2) Describe $\operatorname{ext} B_{\ell^1(\Gamma,\mathbb{C})}$.
(3) Is $\overline{\operatorname{co}\operatorname{ext} B_{\ell^1(\Gamma,\mathbb{F})}}^{\|\cdot\|} = B_{\ell^1(\Gamma,\mathbb{F})}$?

Problem 24. Let K be a compact Hausdorff space.

(1) Describe $\operatorname{ext} B_{\mathcal{C}(K,\mathbb{R})^*}$.
(2) Describe $\overline{\operatorname{co}\operatorname{ext} B_{\mathcal{C}(K,\mathbb{R})^*}}^{\|\cdot\|}$.
(3) Show that $\overline{\operatorname{co}\operatorname{ext} B_{\mathcal{C}(K,\mathbb{R})^*}}^{\|\cdot\|} = B_{\mathcal{C}(K,\mathbb{R})^*}$ without using Krein-Milman theorem.
(4) Solve the problems (1)–(3) for $\mathcal{C}(K,\mathbb{C})$.

`Hint:` *Use the Riesz representation theorem to represent $\mathcal{C}(K,\mathbb{F})^*$ as a space of measures. (1) In the real case show that extreme points are just $\pm\delta_x$, $x \in K$. In the complex case show that extreme points are multiples of Dirac measures by a complex unit. (2) Show that the set consist exactly of measures supported by a countable set. (3) Use (1) an the bipolar theorem.*

Problem 25. Let X be a Banach space

(1) Show that $\overline{\operatorname{co}\operatorname{ext} B_{X^*}}^{w^*} = B_{X^*}$.
(2) Suppose that X is reflexive. Show that $\overline{\operatorname{co}\operatorname{ext} B_{X^*}}^{\|\cdot\|} = B_{X^*}$.
(3) Show by examples that for a nonreflexive space one can have either $\overline{\operatorname{co}\operatorname{ext} B_{X^*}}^{\|\cdot\|} = B_{X^*}$ or $\overline{\operatorname{co}\operatorname{ext} B_{X^*}}^{\|\cdot\|} \subsetneq B_{X^*}$ and both possibilities can take place.

`Hint:` *(1) Combine Krein-Milman and Banach-Alaoglu theorems. (3) Use some of the preceding problems.*

Problem 26. Let K be a compact Hausdorff space.

(1) Describe $\operatorname{ext} B_{\mathcal{C}(K,\mathbb{F})}$.
(2) Deduce that in case K is connected and contains at least two points the space $\mathcal{C}(K,\mathbb{R})$ is not isometric to a dual Banach space.

Problem 27. Show that $\operatorname{ext} B_{L^1([0,1])} = \emptyset$.

Problem 28. Show that $\operatorname{ext} B_{c_0} = \emptyset$.

Problem 29. Let K be a compact convex subset of a HLCS. The point $x \in K$ is called an **exposed point** of K if there is a continuous affine function $f : K \to \mathbb{R}$ such that $f(y) < f(x)$ for each $y \in K \setminus \{x\}$.

(1) Show that any exposed point is also an extreme point.
(2) Show that an extreme point need not be an exposed point.

`Hint:` *(2) Consider the set $K = \operatorname{co}(B((0,0),1) \cup B((1,0),1))$ in $\mathbb{R}^2$.*

Problem 30. Let K be a compact convex subset of a HLCS. A subset $F \subset K$ is called an **exposed face** of K if there is a continuous affine function $f : K \to \mathbb{R}$ such that $F = \{x \in K; f(x) = \max f(K)\}$.

(1) Show that any exposed face of K is also a closed face of K.
(2) Show that a closed face of K need not be an exposed face.
(3) Let F_1 be an exposed face of K and let F_2 be an exposed face of K. Is F_2 necessarily an exposed face of K?

Hint: *(2) Use Problem 29. (3) Consider the example from Problem 29.*

Problem 31. Let K be a compact convex subset of a HLCS and $\mu = \sum_{j=1}^{n} t_j \delta_{x_j}$ a finitely supported probability measure on K (i.e., $x_1, \dots, x_n \in K$, $t_1, \dots, t_n \in [0,1]$, $t_1 + \dots + t_n = 1$). Find the barycenter of μ.

Problem 32. Let $K = [0,1]$ and let λ be the Lebesgue measure on $[0,1]$. Find the barycenter of λ.

Problem 33. Let $K \subset \mathbb{R}^2$ be a nondegenerate triangle with vertices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. Let

$$\mu = \frac{\lambda^2|_K}{\lambda^2(K)},$$

where λ^2 is the two-dimensional Lebesgue measure. Show that the barycenter of μ coincides with the geometric barycenter of the triangle K (i.e., with $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$).

Hint: *Since the Lebesgue measure is invariant with respect to translation and rotation, suppose without loss of generality that $\boldsymbol{a} = (0,0)$, $\boldsymbol{b} = (b,0)$ and $\boldsymbol{c} = (c_1, c_2)$. Then use the definitions and Fubini theorem.*

Problem 34. Let μ be a Borel probability measure on $[0,1]$. Find a formula for its barycenter.

Problem 35. Let $K \subset \mathbb{R}^n$ be a compact convex set and let μ be a Borel probability measure on K. Find a formula for its barycenter.

Hint: *Apply the definition to coordinate projections.*

Problem 36. Let

$$K = \{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 \leq 1 \ \& \ z \in [-1,1]\}.$$

Show that K is a compact convex set and describe $\operatorname{ext} K$.

Problem 37. Let $A = \{(x,y,0) \in \mathbb{R}^3; x^2 + y^2 \leq 1\}$ and $K = \operatorname{co}(A \cup \{(1,0,1), (1,0,-1)\})$. Show that K is a compact convex set, describe $\operatorname{ext} A$ and show that $\operatorname{ext} A$ is not closed.

Problem 38. Let $\varphi : [0,1] \to (0, +\infty)$ be a continuous concave function. Let

$$K = \{(x,y,z) \in \mathbb{R}^3; z \in [0,1] \ \& \ x^2 + \frac{y^2}{\varphi(z)} \leq 1\}.$$

(1) Show that K is a compact convex set.
(2) Describe $\operatorname{ext} K$.
(3) Assume that φ is not affine on $[0,1]$. Show that $\operatorname{ext} K$ is not closed in K.

Hint: *(1) Show that the function $(x,y,z) \mapsto x^2 + \frac{y^2}{\varphi(z)}$ is convex. (2) First show that all the extreme points are on the boundary of K and that the boundary of K is the union of closed discs $D_0 = \{(x,y,0); x^2 + \frac{y^2}{\varphi(0)} \leq 1\}$, $D_1 = \{(x,y,1); x^2 + \frac{y^2}{\varphi(1)} \leq 1\}$ and the set $B = \{(x,y,z) \in \mathbb{R}^3; z \in [0,1] \ \& \ x^2 + \frac{y^2}{\varphi(z)} = 1\}$. Further show that a boundary point of K is not an extreme point of K if and only if it is the center of a nondegenerate segment contained in the boundary of K and that any such segment is contained either in D_0 or in D_1 or in B. Deduce that the extreme points contained in D_0 or D_1 are exactly the points of boundary circles of these discs. Finally suppose that (x_1, y_1, z_1) and (x_2, y_2, z_2) are two different points in B such that the segment connecting them is contained in B. Show that $x_1 = y_1$ and $y_1 = y_2$, so we can suppose we have points (x, y, z_1) and (x, y, z_2). In case $y \neq 0$ show that φ is affine on the interval $[z_1, z_2]$. Summarize* *these results to get a description of* $\operatorname{ext} K$. *(3) If* φ *is not affine, then there is* $z_0 \in (0,1)$ *such that the point* $(z_0, \varphi(z_0))$ *is not the center of any nondegenerate segment on the graph of* φ. *Show that* $(1,0,z_0) \in \overline{\operatorname{ext} K} \setminus \operatorname{ext} K$.

Problem 39. Let $\psi : [0, 2\pi] \to [0, \infty)$ be a bounded upper semicontinuous function such that $\psi(0) = \psi(2\pi)$. (Recall that ψ is **upper semicontinuous** if $\{t; \psi(t) < c\}$ is open for eah $c \in \mathbb{R}$.) Set

$$A = \{(\cos t, \sin t, z); t \in [0, 2\pi] \ \& \ |z| \leq \psi(t)\}$$

and $K = \operatorname{co} A$.

(1) Show that K is a compact convex set.
(2) Describe $\operatorname{ext} K$.
(3) Suppose that $\psi(t) = R(\frac{t}{2\pi})$, where R is the Riemann function, i.e.

$$R(t) = \begin{cases} \frac{1}{q} & \text{if } t = \frac{p}{q} \text{ where } p \in \mathbb{Z},\ q \in \mathbb{N} \text{ and } p, q \text{ are mutually prime,} \\ 0 & \text{if } t \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that $\operatorname{ext} K$ is a G_δ set which is not F_σ.

`Hint:` *(1) Show that* A *is compact by showing it is closed and bounded and use the fact that in a finite-dimensional space the convex hull of a compact set is again compact. (2) Using Proposition XI.21 show that* $\operatorname{ext} K$ *consists exactly of those elements of* A *which are not the center of a nondegenerate segment contained in* A. *(3) Let* $B = \{(\cos t, \sin t, 0); t \in [0, 2\pi]\}$. *Show that* B *is a closed subset of* K *such that both sets* $B \cap \operatorname{ext} K$ *and* $B \setminus \operatorname{ext} K$ *are dense in* B. *Since* $\operatorname{ext} K$ *is* G_δ *by Proposition XI.24(a), use Baire category theorem to show that* $\operatorname{ext} K$ *is not* F_σ.

Problem 40. Let K be a compact Hausdorff space and let $P(K)$ denote the set of all the Radon probability measures on K equipped with the weak* topology inherited from $\mathcal{C}(K)^*$. Denote by $\delta : K \to P(K)$ the mapping assigning to each $x \in K$ the Dirac measure δ_x supported at x. For any $f \in \mathcal{C}(K)$ denote by $\tilde{f}$ the function on $P(K)$ defined by

$$\tilde{f}(\mu) = \int f \, \mathrm{d}\mu, \quad \mu \in P(K).$$

(1) Show that, given $f \in \mathcal{C}(K)$, the function $\tilde{f}$ is a continuous affine function on $P(K)$.
(2) Let $\mu \in P(K)$. Denote by $\delta(\mu)$ the image of μ by the mapping δ. Show that $\delta(\mu)$ is a Radon probability measure on $P(K)$ and its barycenter is μ.
(3) Show that the mapping $f \mapsto \tilde{f}$ is a linear isometry of $\mathcal{C}(K)$ onto the space of all affine continuous functions on $P(K)$ equipped with the sup-norm.

Problem 41. Let K be a compact convex subset of a HLCS and let $f : K \to \mathbb{R}$ be a continuous affine function. Show that f attains its maximum on K at some extreme point of K.

`Hint:` *The set* $\{x \in K; f(x) = \max f(K)\}$ *is a closed face of* K.

Problems to Section XI.4 – compactness, angelicity etc.

Problem 42. Let Γ be an uncountable set and let

$$X = \{x \in [0,1]^\Gamma; \{\gamma \in \Gamma; x(\gamma) \neq 0\} \text{ is countable}\},$$

equipped with the product topology inherited from $[0,1]^\Gamma$.

(1) Show that X is sequentially compact.

(2) Show that X is dense in $[0,1]^\Gamma$.
(3) Deduce that X is not compact.

Problem 43. Let $K = \{0,1\}^{(\{0,1\}^{\mathbb{N}})}$ be equipped with the product topology (where $\{0,1\}$ is considered with the discrete topology).

(1) Define a sequence (x_n) in K by setting

$$x_n(\boldsymbol{a}) = a_n, \quad \boldsymbol{a} = (a_n) \in \{0,1\}^{\mathbb{N}}.$$

Show that the sequence (x_n) has no convergent subsequence.
(2) Deduce that K is a compact space which is not sequentially compact.

Problem 44. Find a countably compact space which is neither compact nor sequentially compact.

`Hint:` *Take the topological sum of preceeding examples.*

Problem 45. Consider the space K and the sequence (x_n) as in Problem 43. Denote $L = \overline{\{x_n; n \in \mathbb{N}\}}$. Further, interpret the index set $\{0,1\}^{\mathbb{N}}$ as the power set $\mathcal{P}(\mathbb{N})$ (identify any subset of $\mathbb{N}$ with its characteristic function). For any element $x \in L$ denote $U(x) = \{A \subset \mathbb{N}; x(A) = 1\}$.

(1) Show that for any $x \in L$ we have
- $A, B \in U(x) \Rightarrow A \cap B \in U(x)$,
- $A \in U(x), \mathbb{N} \supset B \supset A \Rightarrow B \in U(x)$,
- $A \in U(x) \Leftrightarrow \mathbb{N} \setminus A \notin U(x)$.

(2) For $x \in L \setminus \{x_n; n \in \mathbb{N}\}$ show that $U(x)$ contains no finite set.
(3) Show that L contains no one-to-one convergent sequence.

`Hint:` *(1,2) Use the definition of the product topology. (3) Proceed by contradiction. Suppose that (y_k) is a one-to-one convergent sequence in L. Due to Problem 43 we can suppose that the sequence is contained in $L \setminus \{x_n; n \in \mathbb{N}\}$. Using (1) and the assumption that the sequence is one-to-one construct a sequence (A_k) of disjoint infinite subsets of $\mathbb{N}$ such that $A_k \in U(y_k)$ for each $k \in \mathbb{N}$. Let $A = \bigcup\{A_k; k \text{ odd}\}$. Show that the sequence $(y_k(A))$ is not convergent.*

Problem 46. Let X be a topological space such that X^n is Lindelöf for each $n \in \mathbb{N}$. (Recall that a topological space is said to be **Lindelöf** if any its open cover admits a countable subcover.) Show that the space $(\mathcal{C}(X), \tau_p)$ has countable tightness (see Theorem XI.28).

`Hint:` *Show that the proof of Theorem XI.28 works in this more general case.*

Problem 47. Let X be a σ-compact topological space. Show that the space $(\mathcal{C}(X), \tau_p)$ has countable tightness (see Theorem XI.28).

`Hint:` *Use Problem 46.*

Problem 48. Let X be a Hausdorff topological space which can be expressed as the union of a sequence of pairwise disjoint compact sets.

(1) Show that any relatively countably compact subset of $(\mathcal{C}(X), \tau_p)$ is relatively compact.
(2) Show that any separable compact subset of $(\mathcal{C}(X), \tau_p)$ is metrizable.
(3) Deduce that $(\mathcal{C}(X), \tau_p)$ is angelic.

`Hint:` *Let* $X = \bigcup_n K_n$*, where* K_n *are compact and pairwise disjoint. (1) Observe that* $(\mathcal{C}(X), \tau_p)$ *is homeomorphic to the product space* $\prod_n (\mathcal{C}(K_n), \tau_p)$*, use Lemma XI.27 and Tychonoff theorem. (2) Proceed similarly as in the proof of Proposition XI.29 – the statement is clear if* X *is metrizable (which takes place if and only if each* K_n *is metrizable). In the second part of proof do not take just* $\varphi(X)$ *but the topological sum of* $\varphi(K_n)$*,* $n \in \mathbb{N}$*. (3) Combine Problem 47 with (1) and (2), similarly as in the proof of Theorem XI.26.*

Problem 49. Let X and Y be Hausdorff completely regular spaces. Assume X is angelic and there is a continuous one-to-one mapping $f : Y \to X$. Show that Y is angelic as well.

`Hint:` *The key trick is to show the following: If* $A \subset Y$ *is relatively countably compact and* (y_n) *is a sequence in* A *such that the sequence* $(f(y_n))$ *is convergent in* X*, then the sequence* (y_n) *is convergent in* Y*.*

Problem 50. Let X be a σ-compact Hausdorff space. Show that $(\mathcal{C}(X), \tau_p)$ is angelic.

`Hint:` *Let* $X = \bigcup_n K_n$ *with* K_n *compact. Let* Y *be the topological sum of the spaces* K_n*. Find a continuous one-to-one mapping* $F : (\mathcal{C}(X), \tau_p) \to (\mathcal{C}(Y), \tau_p)$ *and use Problems 49 and 48.*

Problem 51. Let X be a metrizable LCS. Show that (X, w) is angelic.

`Hint:` *Use Proposition XI.7(a) and Problems 50 and 49.*

Problem 52. Let X be a normed linear space which is not complete. Show that there is a norm-compact set $K \subset X$ such that $\overline{\operatorname{aco} K}$ is not weakly compact.

`Hint:` *Show that* K *can be taken in the form* $K = \{0\} \cup \{x_n; n \in \mathbb{N}\}$ *where* $x_n \to 0$*. To this end use Proposition XI.11, Theorem XI.12 and Mackey-Arens theorem.*