

X.8 Complements to the theory of unbounded operators

Proposition 43. *Let E be an abstract spectral measure in a Hilbert space H defined on a σ -algebra \mathcal{A} . For an \mathcal{A} -measurable function $f : \mathbb{C} \rightarrow \mathbb{C}$ set $\Phi(f) = \int f dE$.*

- (1) *Let $x \in H$. Set $H_x = \{\Phi(f)x; f \in \mathbb{C}_b(\mathbb{C})\}$. Then H_x is a (not necessarily closed) subspace of H and the mapping $U_x : f \mapsto \Phi(f)x$ is a linear isometry of the space $L^2(E_{x,x})$ onto $\overline{H_x}$.*
- (2) *There exists a set $\Gamma \subset S_H$ satisfying:*
 - $H_x \perp H_y$ for $x, y \in \Gamma, x \neq y$.
 - $\text{span}(\bigcup_{x \in \Gamma} H_x)$ is a dense subspace H .
- (3) *Let $\Omega = \Gamma \times \mathbb{C}$. Let*

$$\tilde{\mathcal{A}} = \{A \subset \Omega; \forall x \in \Gamma : \{\lambda \in \mathbb{C}; (x, \lambda) \in A\} \in \mathcal{A}\}$$

and

$$\mu(A) = \sum_{x \in \Gamma} E_{x,x}(\{\lambda \in \mathbb{C}; (x, \lambda) \in A\}), \quad A \in \tilde{\mathcal{A}}.$$

Then $(\Omega, \tilde{\mathcal{A}}, \mu)$ is a measure space (with a nonnegative measure). Moreover, the mapping $U : L^2(\mu) \rightarrow H$ defined by

$$U(g) = \sum_{x \in \Gamma} \Phi(\lambda \mapsto g(x, \lambda))x, \quad g \in L^2(\mu)$$

is a linear isometry of $L^2(\mu)$ onto H .

- (4) *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an \mathcal{A} -measurable function. Then $\Phi(f) = UM_{\tilde{f}}U^*$, where*

$$\tilde{f}(x, \lambda) = f(\lambda), \quad (x, \lambda) \in \Omega$$

and $M_{\tilde{f}}$ is the operator on $L^2(\mu)$ given by

$$M_{\tilde{f}}g = \tilde{f} \cdot g, \quad g \in D(M_{\tilde{f}}) = \{g \in L^2(\mu); \tilde{f} \cdot g \in L^2(\mu)\}.$$

Theorem 44 (diagonalization of a normal operator). *Let T be a normal operator on a Hilbert space H . Then T is unitarily equivalent to a suitable multiplication operator. I.e., there exist a nonnegative measure μ , a unitary operator $U : L^2(\mu) \rightarrow H$ and a μ -measurable function f such that $T = UM_fU^*$, where M_f is defined as in Proposition 43. Moreover:*

- (a) *If T is selfadjoint, f can be chosen to be real-valued.*
- (b) *If T is bounded, f can be chosen to be bounded.*
- (c) *If H is separable, μ can be chosen to be σ -finite.*

Theorem 45 (an alternative expression of the spectral decomposition of a selfadjoint operator). *Let T be a selfadjoint operator on H and let E be its spectral measure (from Theorem 36). Then $E(\mathbb{C} \setminus \mathbb{R}) = 0$. For $\lambda \in \mathbb{R}$ set $E_\lambda = E((-\infty, \lambda])$. Then:*

- (a) E_λ is an orthogonal projection for each $\lambda \in \mathbb{R}$.
- (b) $E_\lambda E_\mu = E_\mu E_\lambda = E_{\min\{\lambda, \mu\}}$ for $\lambda, \mu \in \mathbb{R}$.
- (c) $\lim_{\mu \rightarrow \lambda+} E_\mu x = E_\lambda x$ for each $x \in H$ and $\lambda \in \mathbb{R}$.
- (d) If λ is not an eigenvalue of T , then $\lim_{\mu \rightarrow \lambda-} E_\mu x = E_\lambda x$ for each $x \in H$.
- (e) If λ is an eigenvalue of T , then the formula $P_\lambda x = \lim_{\mu \rightarrow \lambda-} E_\mu x$, $x \in H$, defines an orthogonal projection such that $E_\lambda - P_\lambda$ is also an orthogonal projection and, moreover, $R(E_\lambda - P_\lambda) = \text{Ker}(\lambda I - T)$.
- (f) $\lim_{\mu \rightarrow -\infty} E_\mu x = 0$ and $\lim_{\mu \rightarrow +\infty} E_\mu x = x$ for each $x \in H$.
- (g) A real number λ belongs to $\rho(T)$ if and only if the mapping $\mu \mapsto E_\mu$ is constant on a neighborhood of λ .

Theorem 46 (selfadjoint operators on a real Hilbert space). *Let H be a real Hilbert space and let T be an operator on H . Then T^* can be defined in the same way as in the complex case (see Section X.3). Let H_C be the hilbertian complexification of H , i.e., the space $H_C = H + iH = \{x + iy; x, y \in H\}$ equipped with the scalar product*

$$\langle x + iy, u + iv \rangle = \langle x, u \rangle + \langle y, v \rangle + i \langle y, u \rangle - i \langle x, v \rangle, \quad x + iy, u + iv \in H_C.$$

Define an operator T_C on H_C by

$$T_C(x + iy) = T(x) + iT(y), \quad x + iy \in D(T_C) = D(T) + iD(T).$$

Then:

- (a) If T is densely defined, then T_C is also densely defined and $(T_C)^* = (T^*)_C$.
- (b) If T is selfadjoint, then T_C is also selfadjoint and, moreover, for $\lambda \in \mathbb{R}$ we have

$$\lambda I - T \text{ is invertible in } L(H) \Leftrightarrow \lambda I_C - T_C \text{ is invertible in } L(H_C).$$

- (c) Let T be selfadjoint and let E be the spectral measure of T_C , let \mathcal{A} be the corresponding σ -algebra. Then the formula

$$E_R(A) = E(A)|_H, \quad A \in \mathcal{A}$$

defines a “real spectral measure” on \mathbb{R} and $T = \int \text{id} \, dE_R$.