

X.7 Normal unbounded operators

Definition. A densely defined closed operator T on a Hilbert space is called **normal** if $T^*T = TT^*$.

Lemma 38 (on T^*T). *Let T be a closed densely defined operator on H . Then:*

- (a) $I + T^*T$ is a bijection of $D(T^*T)$ onto H .
- (b) Denote by B the inverse operator to $I + T^*T$ and $C = TB$. Then B and C belong to $L(H)$ and their norms are at most 1. Moreover, B is positive.
- (c) T^*T is selfadjoint and T is the closure of $T|_{D(T^*T)}$.

Lemma 39. *Let T be a normal operator on H . Then:*

- (a) $D(T) = D(T^*)$
- (b) $\|Tx\| = \|T^*x\|$ for $x \in D(T)$.
- (c) If $S \supset T$ is normal, then $S = T$.

Theorem 40 (spectral decomposition of an unbounded normal operator). *If T is a normal operator on H , then there exists a unique abstract spectral measure E in H such that $T = \int \text{id} \, dE$. This measure can be described as follows: Let B be the operator from Lemma 38. For $j \in \mathbb{N}$ let $P_j = \chi_{(\frac{1}{j+1}, \frac{1}{j}]}(B)$. Then TP_j is a bounded normal operator, let E^j be its spectral measure and let \mathcal{A}_j be the corresponding σ -algebra. Then the sought measure E is given by*

$$E(A)x = \sum_{j=1}^{\infty} E^j(A)P_jx, \quad x \in H, A \in \mathcal{A} = \bigcap_{j \in \mathbb{N}} \mathcal{A}_j.$$

Corollary 41. *Let T be a normal operator on H . Then T is bounded if and only if $\sigma(T)$ is a bounded set.*

Corollary 42. *Let E be an abstract spectral measure in a Hilbert space H defined on a σ -algebra \mathcal{A} . Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a \mathcal{A} -measurable function and let $T = \int f \, dE$. Then T is a normal operator and its spectral measure (i.e., the measure from Theorem 40) is the image of E under f (in the sense of Lemma 35).*