X.6 Spectral decomposition of a selfadjoint operator

Reminder: Let $T \in L(H)$ be a normal operator. Let

$$f \mapsto \tilde{f}(T), \quad f \in C(\sigma(T)),$$

denote the continuous functional calculus (see Section VIII.6). For any pair $x, y \in H$ let $E_{x,y}$ be the Radon measure on $\sigma(T)$ satisfying

$$\left\langle \tilde{f}(T)x, y \right\rangle = \int_{\sigma(T)} f \,\mathrm{d}E_{x,y}\,, \quad f \in C(\sigma(T)).$$

Then:

(1) $\|E_{x,y}\| \le \|x\| \cdot \|y\|$ for $x, y \in H$,
(2) $E_{x,x} \ge 0$ for each $x \in H$,
(3) the mapping $x \mapsto E_{x,y}$ is linear for each $y \in H$,
(4) the mapping $y \mapsto E_{x,y}$ is conjugate linear for each $x \in H$.

Let $\mathcal{A}_T$ be the σ-algebra of those subsets of $\sigma(T)$, which are $E_{x,y}$-measurable for any $x, y \in H$. If $h : \sigma(T) \to \mathbb{C}$ is a bounded $\mathcal{A}_T$-measurable function, by $\tilde{h}(T)$ we denote the unique element of $L(H)$ satisfying

$$\left\langle \tilde{h}(T)x, y \right\rangle = \int_{\sigma(T)} h \,\mathrm{d}E_{x,y}.$$

The mapping $h \mapsto \tilde{h}(T)$ is called the **measurable calculus** of the operator T. For $A \in \mathcal{A}_T$ denote

$$E_T(A) = \widetilde{\chi_A}(T).$$

Then $E_T(A)$ is an orthogonal projection. The mapping $A \mapsto E_T(A)$ is called the **spectral measure** of the operator T.

We extend the mapping E_T to the σ-algebra $\{A \subset \mathbb{C}; A \cap \sigma(T) \in \mathcal{A}_T\}$ of subsets of $\mathbb{C}$ by $E_T(A) = E_T(A \cap \sigma(T))$.

Proposition 31. *Let $T \in L(H)$ be a normal operator. Then E_T is an abstract spectral measure.*

Proposition 32. *Let E be an abstract spectral measure defined on a σ-algebra $\mathcal{A}$, $g \in L^\infty(E)$ and $T = \int g \,\mathrm{d}E$. Th the spectral measure of the operator T is given by the formula*

$$E_T(A) = E(g^{-1}(A)) \text{ for } A \in \mathcal{A}_T = \{A \subset \mathbb{C}; g^{-1}(A) \in \mathcal{A}\}.$$

Theorem 33 (spectral decomposition of a bounded normal operator). *Let $T \in L(H)$ be a normal operator. Then:*

(a) $T = \int \mathrm{id}\, \mathrm{d}E_T$. *Moreover, the unique abstract spectral measure E satisfying $T = \int \mathrm{id}\, \mathrm{d}E$ is E_T.*
(b) *If g is a bounded $\mathcal{A}$-measurable function, then $\tilde{g}(T) = \int g \, \mathrm{d}E_T$.*

Lemma 34. *Let T be a selfadjoint operator on H. Let E be the spectral measure of the operator C_T. Then*

$$T = \int i \frac{1+z}{1-z} \, \mathrm{d}E(z).$$

Lemma 35 (on the image of a spectral measure). *Let F be an abstract spectral measure in H defined on a σ-algebra $\mathcal{A}$ and let $\varphi : \mathbb{C} \to \mathbb{C}$ be an $\mathcal{A}$-measurable mapping. Define*

$$\mathcal{A}' = \{A \subset \mathbb{C} : \varphi^{-1}(A) \in \mathcal{A}\}$$

and for $A \in \mathcal{A}'$ set

$$E(A) = F(\varphi^{-1}(A)).$$

Then E is an abstract spectral measure in H and for each $\mathcal{A}'$-measurable function f one has

$$\int f \, \mathrm{d}E = \int f \circ \varphi \, \mathrm{d}F.$$

Theorem 36 (spectral decomposition of a selfadjoint operator). *If T is a selfadjoint operator on a Hilbert space H, then there exists a unique abstract spectral measure E in H such that $T = \int \mathrm{id}\, \mathrm{d}E$.*

This measure E is the image of the spectral measure of the operator C_T under the Borel mapping $z \mapsto i\frac{1+z}{1-z}$.

Corollary 37. *Let T be a selfadjoint operator on H. Then T is bounded if and only if $\sigma(T)$ is a bounded set.*