

## X.4 Symmetric operators and Cayley transform

**Definition.** Let  $S$  be a symmetric (not necessarily densely defined) operator on  $H$ . Denote by  $C_S$  the operator

$$C_S = (S - iI)(S + iI)^{-1}.$$

Then  $C_S$  is an operator on  $H$ , which is called the **Cayley transform of the operator  $S$** .

**Theorem 21** (properties of  $C_S$ ). *Let  $S$  be a symmetric operator on  $H$  and let  $C_S$  be its Cayley transform. Then*

- (a)  $C_S$  is a linear isometry of  $D(C_S) = R(S + iI)$  onto  $R(C_S) = R(S - iI)$ .
- (b)  $I - C_S = 2i(S + iI)^{-1}$ ; in particular, the operator  $I - C_S$  is one-to-one and  $R(I - C_S) = D(S)$ .
- (c)  $S = i(I + C_S)(I - C_S)^{-1}$ .
- (d)  $C_S$  is closed  $\Leftrightarrow S$  is closed  $\Leftrightarrow D(C_S)$  is closed  $\Leftrightarrow R(C_S)$  is closed.

**Lemma 22** (on an isometric operator). *Let  $U$  be any operator on  $H$ , which is an isometry of  $D(U)$  onto  $R(U)$ . Then*

- (a)  $\langle Ux, Uy \rangle = \langle x, y \rangle$  for any  $x, y \in D(U)$ . In particular:  $U$  is unitary if and only if  $D(U) = R(U) = H$ .
- (b)  $\text{Ker}(I - U) = D(U) \cap (R(I - U))^\perp$ . In particular, if  $R(I - U)$  is dense in  $H$ , then  $I - U$  is one-to-one.

**Theorem 23** (range of the Cayley transform). *Let  $U$  be an operator on  $H$ , which is an isometry of  $D(U)$  onto  $R(U)$ . Suppose that  $I - U$  is one-to-one. Then the operator  $S = i(I + U)(I - U)^{-1}$  is symmetric and  $C_S = U$ . Further,  $S$  is densely defined if and only if  $R(I - U)$  is dense.*

**Theorem 24** (Cayley transform for selfadjoint operators).

- (a) Let  $S$  be a symmetric operator on  $H$ . Then  $S$  is selfadjoint if and only if  $C_S$  is a unitary operator.
- (b) Let  $U$  be a unitary operator on  $H$  such that  $I - U$  is one-to-one. Then the operator  $S = i(I + U)(I - U)^{-1}$  is selfadjoint and  $C_S = U$ .

**Remarks.**

- (1) Let  $S$  and  $T$  be symmetric operators on  $H$ . Then  $S \subset T$  if and only if  $C_S \subset C_T$ .
- (2) Let  $S$  be a densely defined closed symmetric operator on  $H$ . The codimensions of the subspaces  $D(C_S)$  and  $R(C_S)$  (i.e., the dimensions of their orthogonal complements) are called the **deficiency indices** of the operator  $T$ . Then:
  - $T$  is selfadjoint if and only if both deficiency indices are zero.
  - $T$  is a maximal symmetric operator if and only if at least one of the deficiency indices is zero.
  - $T$  has a selfadjoint extension if and only if both deficiency indices are the same (i.e., if and only if there exists a linear isometry of  $(D(C_S))^\perp$  onto  $(R(C_S))^\perp$ ).
- (3) Let  $S$  be a closed symmetric operator on  $H$ , not necessarily densely defined. Also in this case one may define the deficiency indices and the above mentioned equivalences do still hold, only the proof is a bit more complicated. Moreover:
  - If  $D(C_S) = H$  or  $R(C_S) = H$ , then  $S$  is densely defined.