X.3 Operators on a Hilbert space

Convention: In the sequel we will consider only operators on a complex Hilbert space H. The inner product of $x, y \in H$ is denoted by $\langle x, y\rangle$.

Remark: If H is a Hilbert space, then $H \times H$ is also a Hilbert space, if we define the inner product by

$$\langle (x_1, x_2), (y_1, y_2)\rangle = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle, \qquad (x_1, x_2), (y_1, y_2) \in H \times H.$$

Definition. Let T be a densely defined operator on H.

- By $D(T^*)$ we denote the set of those $y \in H$, for which the mapping
$$x \mapsto \langle Tx, y\rangle$$
is continuous on $D(T)$.
- For $y \in D(T^*)$ denote by $T^* y$ the unique element of H satisfying
$$\langle Tx, y\rangle = \langle x, T^* y\rangle \text{ for each } x \in D(T).$$

Lemma 10. *Let T be a densely defined operator on H. Then $D(T^*)$ is a linear subspace of H and T^* is an operator on H with domain $D(T^*)$.*

Remark. Let T be an operator on H, which is not densely defined. Set $K = \overline{D(T)}$. The definition of $D(T^*)$ still makes sense. Moreover, for each $y \in D(T^*)$ there exists a unique $z \in K$ satisfying $\langle Tx, y\rangle = \langle x, z\rangle$ for $x \in D(T)$. It would be possible to define T^* as an operator from H to K (which is a special case of operators on H). If we, moreover, denote by P the orthogonal projection of H onto K, then PT is a densely defined operator on K, $D((PT)^*) = D(T^*) \cap K$ and $(PT)^*$ is the restriction of the operator T^* from the previous sentence to $D((PT)^*)$.

Definition. The operator T^* is said to be the **adjoint operator to** T.

Proposition 11 (properties of adjoint operator).

(a) *If S is densely defined and $S \subset T$, then $T^* \subset S^*$.*
(b) *If $S + T$ is densely defined, then $S^* + T^* \subset (S + T)^*$. If moreover $S \in L(H)$, then $S^* + T^* = (S + T)^*$.*
(c) *If S and ST are densely defined, then $T^*S^* \subset (ST)^*$. If moreover $S \in L(H)$, then $T^*S^* = (ST)^*$.*

Proposition 12 (on kernel and range). *For a densely defined operator T one has* $\operatorname{Ker}(T^*) = R(T)^\perp$.

Lemma 13 (on the transformation of a graph). *Define $V : H \times H \to H \times H$ by $V(x, y) = (-y, x)$. Then*

(a) *V is a unitary operator on $H \times H$,*
(b) *$G(T^*) = (V(G(T)))^\perp = V(G(T)^\perp)$ for a densely defined operator T on H.*

Remark: Lemma 13 is a very useful tool for working with adjoint operators. The assertion (b) is a brief expression of the equivalence

$$z = T^* y \Leftrightarrow (\forall x \in D(T) : (y, z) \perp (-Tx, x)) \Leftrightarrow (\forall x \in D(T) : \langle x, z\rangle = \langle Tx, y\rangle).$$

Lemma 14. *Let T be densely defined, one-to-one and let $R(T)$ be dense. Then $(T^{-1})^* = (T^*)^{-1}$.*

Proposition 15 (adjoint operator and closedness). *Let T be densely defined. Then:*

(a) *The operator T^* is closed.*
(b) *T has a closed extension if and only if T^* is densely defined (then $\overline{T} = T^{**}$).*
(c) *T is closed if and only if $T = T^{**}$ (implicitly T^* is densely defined).*

Definition. Let T be an operator on H.

- We say that T is **symmetric** if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for each $x, y \in D(T)$.
- We say that T is **selfadjoint** if $T = T^*$.

Remarks.

(1) A symmetric operator need not be densely defined. If T is densely defined, then T is symmetric if and only if $T \subset T^*$.
(2) Let T be an operator on H, which is not densely defined. Set $K = \overline{D(T)}$ and let P be the orthogonal projection on K. Then PT is a densely defined operator on K. Moreover, T is symmetric if and only if PT je symmetric.
(3) A selfadjoint operator is always densely defined (in order T^* is defined) and closed (by Proposition 15(a)).

Lemma 16. *Let T be a selfadjoint operator. Then T is maximal symmetric (i.e., there is no proper symmetric extension of T).*

Remark. A densely defined maximal symmetric operator need not be selfadjoint. This follows from the remarks at the end of Section X.4.

Proposition 17 (further properties of symmetric operators). *Let T be a symmetric densely defined operator on H. Then:*

(a) *$\overline{T}$ is symmetric.*
(b) *If $D(T) = H$, then T is bounded and selfadjoint.*
(c) *If $R(T)$ is dense, then T is one-to-one.*
(d) *If $R(T) = H$, then T is one-to-one, selfadjoint and $T^{-1} \in L(H)$.*
(e) *If T is selfadjoint and one-to-one, then T^{-1} is selfadjoint (in particular densely defined).*

Lemma 18 (on $(\alpha + i\beta)I - S$). *Let S be a symmetric operator on H and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then $\lambda I - S$ is one-to-one and its inverse is continuous on $R(\lambda I - S)$. Moreover, S is closed if and only if $R(\lambda I - S)$ is closed.*

Theorem 19 (spectrum of a selfadjoint operator). *For each selfadjoint operator T one has $\emptyset \neq \sigma(T) \subset \mathbb{R}$.*

Corollary 20 (characterization of selfadjoint operators among symmetric ones). *For a densely defined operator T on H the following assertions are equivalent:*

(i) *T is selfadjoint;*
(ii) *T is symmetric and $\sigma(T) \subset \mathbb{R}$;*
(iii) *T is symmetric and there exists $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $\lambda, \overline{\lambda} \in \rho(T)$.*