XI.2 bw^*-topology and Krein-Šmulyan theorem

Definition. Let X be normed linear space. We say that a set $A \subset X^*$ is bw^*-**open** if for each $r > 0$ the set $A \cap rB_{X^*}$ is (relatively) w^*-open in rB_{X^*}.

Lemma 10. *Let X be a normed linear space. Then the family of all the bw^*-open subsets X^* is a topology, which is finer than the w^*-topology.*

Definition. Let X be a normed linear space. Then the family of all the bw^*-open subsets X^* is called the bw^*-**topology**.

Proposition 11. *Let X be a normed linear space. The bw^*-topology on X^* coincides with the topology of uniform convergence on sequences in X, which are norm-convergent to zero.*

Theorem 12 (Banach-Dieudonné). *Let X be a normed linear space and let $\varkappa : X \to X^{**}$ denote the canonical embedding. Then*

$$(X^*, bw^*)^* = \overline{\varkappa(X)}.$$

In other words, the dual to (X^*, bw^*) can be identified with the completion of X. In particular,

$$(X^*, bw^*)^* = \varkappa(X) \iff X \text{ is complete.}$$

Corollary 13 (Krein-Šmulyan). *Let X be a Banach space and let $A \subset X^*$ be a convex set. Then*

$$A \text{ is } w^*\text{-closed} \iff \forall r > 0 : A \cap rB_{X^*} \text{ is } w^*\text{-closed.}$$

Corollary 14 (Banach-Dieudonné). *Let X be a Banach space and let f be a linear functional on X^* (i.e., $f \in (X^*)^\#$. Then*

$$f \in \varkappa(X) \iff f|_{B_{X^*}} \text{ is } w^*\text{-continuous.}$$

Theorem 15. *Let X be a Banach space. Denote $K = (B_{X^*}, w^*)$. Then K is a compact Hausdorff space. Define the mapping $J : X \to \mathcal{C}(K)$ by $J(x) = \varkappa(x)|_K$, $x \in X$. Then J is a linear isometry of X into $\mathcal{C}(K)$, a homeomorphism of (X, w) into $(\mathcal{C}(K), \tau_p)$ and, moreover, $J(X)$ is τ_p-closed in $\mathcal{C}(K)$.*

Remarks. Let X be a Banach space and $A \subset X^*$ a convex set. Then $\overline{A}^{w^*} = \overline{A}^{bw^*}$. However, it may happen that

$$\bigcup_{r>0} \overline{A \cap rB_{X^*}}^{w^*} \subsetneq \overline{A}^{w^*},$$

even if A is a subspace. This is illustrated by distingushing the following cases:

Let $Y \subset\subset X^*$. Define the seminorm $\tilde{q}_Y$ on X by

$$\tilde{q}_Y(x) = \sup\{|f(x)|\,; f \in Y \ \&\ \|f\| \le 1\},$$

i.e., $\tilde{q}_Y = q_{B_{X^*} \cap Y}$. Then the following hold:

1. $\tilde{q}_Y$ is a norm on $X \iff \overline{Y}^{w^*} = X^*$.
2. $\tilde{q}_Y = \|\cdot\| \iff \overline{Y \cap B_{X^*}}^{w^*} = B_{X^*}$. In this Y is said to be a 1-**norming** subspace X^*.
3. $\tilde{q}_Y$ is an equivalent norm on $X \iff \exists r > 0 : \overline{Y \cap B_{X^*}}^{w^*} \supset \frac{1}{r} B_{X^*}$ $\iff \bigcup_{r>0} \overline{Y \cap rB_{X^*}}^{w^*} = X^*$. In this Y is said to be a **norming** (or, more precisely, r-**norming**, where r is the number from the second condition) subspace of X^*.