

Proof of Proposition X. 43

Let E be an abstract spectral measure on H ,
 \mathcal{A} the domain σ -algebra
 for $f: \mathcal{A} \rightarrow \mathbb{C}$ \mathcal{A} -measurable set $\Phi(f) = \int f dE$

(1) For $x \in H$ set $H_x := \{ \Phi(f)x ; f \in C_b(\mathcal{A}) \}$

Then

- $H_x \subset H$ (as Φ is linear), not necess. closed
- $f \in C_b(\mathcal{A}) \Rightarrow \| \Phi(f)x \| = \| f \|_{L^2(E_{x,x})}$ (Th 27(d))

For $f \in L^2(E_{x,x})$ define $U_x(f) = \Phi(f)x$

The U_x is a linear isometry of $L^2(E_{x,x})$ into H
 (by Th 28)

Moreover, the range of U_x is $\overline{H_x}$ (cts functions are dense in $L^2(E_{x,x})$)

(2) If $y \perp H_x$, then $H_y \perp H_x$ (here $\overline{H_y} \perp \overline{H_x}$)

Let $y \in \perp H_x$. Fix $f, g \in C_b(\mathcal{A})$. Then

$$\begin{aligned} \langle \Phi(g)y, \Phi(f)x \rangle &= \langle y, \Phi(g)^* \Phi(f)x \rangle \stackrel{\text{Th 27(a)}}{=} \\ &= \langle y, \underbrace{\Phi(\bar{g} \cdot f)x}_{\in H_x} \rangle = 0 \end{aligned}$$

(3) Let $\mathcal{N} \subset S_H$ be maximal s.t. $H_x \perp H_y, \forall y \in \mathcal{N}, \forall y$
 (It exists by Zorn's Lemma)

Then $\text{span} \left(\bigcup_{y \in \mathcal{N}} H_y \right)$ is dense in H [otherwise $\exists z \in S_H$
 $z \perp \bigcup_{y \in \mathcal{N}} H_y \Rightarrow z$
 can be added to \mathcal{N}]

(4) Let $\Omega = \Gamma \times \mathbb{C}$ Define a σ -algebra $\tilde{\mathcal{A}}$ on Ω by

$$\tilde{\mathcal{A}} = \{ A \subset \Omega; \forall x \in \Gamma: \{ \lambda \in \mathbb{C}; (x, \lambda) \in A \} \in \mathcal{A} \}$$

The $\tilde{\mathcal{A}}$ is clearly a σ -algebra. For $A \in \tilde{\mathcal{A}}$ set

$$\mu(A) = \sum_{x \in \Gamma} E_{x,x} \{ \lambda \in \mathbb{C}, (x, \lambda) \in A \}$$

Then μ is a nonnegative σ -additive measure on Ω

(5) Define $U: L^2(\mu) \rightarrow H$ by

$$U(g) = \sum_{x \in \Gamma} \Phi_x(\lambda \mapsto g(x, \lambda))_+, \quad g \in L^2(\mu)$$

Then U is a linear isometry of $L^2(\mu)$ onto H :

• Fix $g \in L^2(\mu)$. For $x \in \Gamma$ set $g_x(\lambda) = g(x, \lambda), \lambda \in \mathbb{C}$

Then $g_x \in L^2(E_{x,x})$. Moreover,

$$\text{by (1)} \quad \Phi_x(g_x)_+ \in \overline{H_x}, \quad \|\Phi_x(g_x)_+\| = \|g_x\|_{L^2(E_{x,x})}$$

$$\bullet \text{ Further } \|g\|_{L^2(\mu)} = \left(\sum_{x \in \Gamma} \|g_x\|_{L^2(E_{x,x})}^2 \right)^{1/2}$$

Since $\overline{H_x} \perp \overline{H_y}$ for $x \neq y$, we conclude

$$\|U(g)\| = \|g\|_{L^2(\mu)}$$

Since U is clearly linear, we deduce that U is a linear isometry

• U is onto: $z \in H \Rightarrow z = \sum_{x \in \Gamma} z_x$ for some $z_x \in \overline{H_{x,x}}$

Since U_x is onto $\dots \exists f_x \in L^2(E_{x,x}), U_x(f_x) = z_x, \dots$
 $f(x, \lambda) = f_x(\lambda), (x, \lambda) \in \Omega \Rightarrow f \in L^2(\mu), U(f) = z$

(6) Let $f: \Omega \rightarrow \mathbb{C}$ be \mathcal{A} -measurable
 Set $\tilde{f}(A, \omega) = f(\omega)$, $(A, \omega) \in \Omega \Rightarrow \tilde{f}$ is $\tilde{\mathcal{A}}$ -measurable

We claim that $\Phi(f) = U M_f U^*$, i.e. $M_f = U^* \Phi(f) U$

Recall $M_f(g) = \tilde{f}g$, $g \in D(M_f) = \{g, g \in C^2(\mu); \tilde{f} \cdot g \in C^2(\mu)\}$

$\bullet g \in D(M_f) \Leftrightarrow g \cdot \tilde{f} \in L^2(\mu) \Leftrightarrow \sum_{x \in \Omega} \int |g(x, \omega) \cdot f(\omega)|^2 < \infty$

$\bullet D(U^* \Phi(f) U) = \{g \in D(U); U_g \in D(\Phi(f))\}$, $D(U) = L^2(\mu)$

so: $g \in D(U^* \Phi(f) U) \Leftrightarrow U_g \in D(\Phi(f)) \Leftrightarrow f \in L^2(E_{U(g), U(g)}) \Leftrightarrow$

$\Gamma_{E_{U(g), U(g)}}(A) = \langle E(A) U(g), U(g) \rangle = \langle E(A) U(g), \sum_{x \in \Omega} U_x g_x \rangle$

$= \sum_{x \in \Omega} \langle E(A) U(g), U_x g_x \rangle = \sum_{x, y \in \Omega} \langle E(A) U_y g_y, U_x g_x \rangle =$

$= \sum_{x, y \in \Omega} \langle \Phi(\chi_A) \Phi(g_y)_y, \Phi(g_x)_x \rangle =$

$= \sum_{x, y \in \Omega} \langle \Phi(\chi_A g_y)_y, \Phi(g_x)_x \rangle \stackrel{H_x \perp H_y}{=} \sum_{x \in \Omega} \langle \Phi(\chi_A g_x)_x, \Phi(g_x)_x \rangle$

$= \sum_{x \in \Omega} \langle E(A) \Phi(g_x)_x, \Phi(g_x)_x \rangle \stackrel{29(c), (5)}{=} \sum_{x \in \Omega} \langle \Phi(\chi_A g_x)_x, \Phi(g_x)_x \rangle$

$= \sum_{x \in \Omega} \int_A |g_x|^2 dE_{x, x} = \int_{\Omega \times A} |g|^2 d\mu$

$\Leftrightarrow \int |f|^2 d E_{U(g), U(g)} < \infty \Leftrightarrow \int |f|^2 |g|^2 d\mu < \infty$

$\Leftrightarrow \tilde{f} \cdot g \in L^2(\mu)$

$$S_0, D(M_f^\sim) = D(U^* \Phi(f) U)$$

Moreover, if g, h are common domain, then

$$\langle U^* \Phi(f) U g, h \rangle = \langle \Phi(f) U g, U h \rangle =$$

$$= \sum_{x \in \mathcal{P}} \langle \Phi(f) \Phi(g_x)_x, \Phi(h_y)_y \rangle \stackrel{7.4.27(b)}{=} \sum_{x \in \mathcal{P}} \langle \Phi(fg_x)_x, \Phi(h_x)_x \rangle$$

$$\mathcal{H}_+ \perp \mathcal{H}_\perp \text{ for } x \neq y$$

$$= \sum_{x \in \mathcal{P}} \langle \Phi(fg_x)_x, \Phi(h_x)_x \rangle = \sum_{x \in \mathcal{P}} \langle \Phi(fg_x)_x, \Phi(h_x)_x \rangle$$

$$= \sum_{x \in \mathcal{P}} \lim_{n \rightarrow \infty} \langle \Phi_0((fg_x)_n)_x, \Phi_0(h_x)_x \rangle =$$

$$= \sum_{x \in \mathcal{P}} \lim_{n \rightarrow \infty} \langle \Phi_0((h_x f g_x)_n)_x, \Phi_0(h_x)_x \rangle = \sum_{x \in \mathcal{P}} \langle \Phi_0(h_x f g_x)_x, \Phi_0(h_x)_x \rangle$$

$$= \sum_{x \in \mathcal{P}} \lim_{n \rightarrow \infty} \int (h_x f g_x)_n dE_{x,+} = \sum_{x \in \mathcal{P}} \int h_x f g_x dE_{x,+}$$

$$= \int h \widehat{f} g d\mu = \langle M_f^\sim g, h \rangle$$

Proof of Theorem X.4.4

Let T be a normal operator on H .

Let $E = E_T$ be its spectral measure.

Let μ and $U: L^2(\mu) \rightarrow H$ be as in Prop. 43.

Then $T = U M_{\tilde{cd}} U^*$, using the notation from Prop. 43,

$$\text{as } T = \int cd \, dE$$

Moreover:

- T self-adjoint $\Rightarrow cd$ is essentially real-valued, so \tilde{cd} is also essentially real-valued.
- T self-adjoint $\Rightarrow cd$ is essentially self-adjoint $\Rightarrow \tilde{cd}$ is essentially self-adjoint.
- H separable $\Rightarrow \mu$ can be chosen to be σ -finite, as to do so P from the proof of Prop. 43 is countable.