Theorem X.28 Let E be an abstract spectral measure in H, defined on $\mathcal{A}$, $f : \mathbb{C} \to \mathbb{C}$ $\mathcal{A}$-measurable function.
Set $D(\Phi(f)) := \{x \in H ; \int |f|^2 dE_{x,x} < \infty\}$
Then $D(\Phi(f))$ is a dense linear subspace of H.
Further, $\exists!\ \Phi(f)$, an operator on H with domain $D(\Phi(f))$
s.t. $\quad \langle \Phi(f)x, y\rangle = \int f \, dE_{x,y}$, $x, y \in D(\Phi(f))$.

Moreover, $\|\Phi(f)x\| = \left(\int |f|^2 dE_{x,x}\right)^{1/2}$, $x \in D(\Phi(f))$.

Proof ① $D(\Phi(f))$ is a linear subspace of H:
- clearly $0 \in D(\Phi(f))$, as $E_{0,0} = 0$
- $x \in D(\Phi(f))$, $\alpha \in \mathbb{C} \Rightarrow \alpha x \in D(\Phi(f))$
  as $E_{\alpha x, \alpha x} = |\alpha|^2 E_{x,x}$ (by L25(a,b))
- $x, y \in D(\Phi(f)) \Rightarrow x + y \in D(\Phi(f))$
  as $E_{x+y, x+y} \leq 2(E_{x,x} + E_{y,y})$ (by L25(g))

② For $n \in \mathbb{N}$ set $A_n := \{\lambda \in \mathbb{C},\ |f(\lambda)| \leq n\}$
Then $A_n \in \mathcal{A}$. Moreover, $R(E(A_n)) \subset D(\Phi(f))$

$\Big\lceil$ $x \in R(E(A_n))$, i.e. $\underbrace{E(A_n)x = x}$

Then $E_{x,x}(A_n) = \langle E(A_n)x, x\rangle = \langle x, x\rangle = \langle Ix, x\rangle$
$= \langle E(\mathbb{C})x, x\rangle = E_{x,x}(\mathbb{C})$

So, $E_{x,x}(\mathbb{C} \setminus A_n) = 0$, in other words $|f| \leq n$ $E_{x,x}$-a.e.

So, $\int |f|^2 dE_{x,x} \leq \int n^2 dE_{x,x} < \infty$

Hence, $x \in D(\Phi(f))$ $\Big\rfloor$

③ $\forall x \in H$: $E(A_n)x \to x$, hence $D(\Phi(f))$ is dense in H

$\Big\lceil$ $\|x - E(A_n)x\|^2 \overset{(iii)}{=} \|E(\mathbb{C})x - E(A_n)x\|^2 \overset{(vi)}{=}$

$= \|E(\mathbb{C}\setminus A_n)x\|^2 = \langle E(\mathbb{C}\setminus A_n)x, E(\mathbb{C}\setminus A_n)x\rangle =$

$= \langle E(\mathbb{C}\setminus A_n)x, x\rangle = E_{x,x}(\mathbb{C}\setminus A_n) \xrightarrow{n\to\infty} 0$

$E(\mathbb{C}\setminus A_n)$ is an OG projection

$\mathbb{C}\setminus A_n \Downarrow$, $\bigcap_n (\mathbb{C}\setminus A_n) = \emptyset$

$E_{x,x}$ is a finite measure $\Big\rfloor$

④ $x, y \in D(\Phi(f)) \Rightarrow \int f \, dE_{x,y}$ is well defined

$\Big\lceil$ By Lemma 25 (f): $|E_{x,y}(A)| \le \frac{1}{2}(E_{x,x}(A) + E_{y,y}(A))$, $A \in \mathcal{A}$,

so by the definition of absolute variation: $|E_{x,y}| \le \frac{1}{2}(E_{x,x} + E_{y,y})$

Since $x, y \in D(\Phi(f))$ $\Rightarrow$ $\left.\begin{array}{l} f \in \mathcal{L}^2(E_{x,x}) \subset \mathcal{L}^1(E_{x,x}) \\ f \in \mathcal{L}^2(E_{y,y}) \subset \mathcal{L}^1(E_{y,y}) \end{array}\right\} \Rightarrow f \in \mathcal{L}^1(|E_{x,y}|)$

$E_{x,x}, E_{y,y}$ finite measures $\Big\rfloor$

⑤ Set $f_n := f \cdot \chi_{A_n}$ (A_n defined in ②) above

Then f_n is bdd $\mathcal{A}$-measurable $\Rightarrow$ we have $\Phi_0(f_n) \in \mathcal{L}(H)$ by Thm 27.

For $x \in D(\Phi(f))$ set $\Phi(f)x := \lim_{n\to\infty} \Phi_0(f_n)x$.

The limit exists, as the sequence $(\Phi_0(f_n)x)_{n=1}^{\infty}$ is Cauchy:

$$m < n \Rightarrow \quad \|\Phi_0(f_n)x - \Phi_0(f_m)x\|^2 \overset{\text{linearity of } \Phi_0}{=} \|\Phi_0(f_n - f_m)x\|^2 =$$

$$\overset{V\,27(d)}{=} \int |f_n - f_m|^2 \, d\hat{E}_{x,x} = \int_{A_n \setminus A_m} |f|^2 \, d\hat{E}_{x,x} \le \int_{\mathbb{C} \setminus A_m} |f|^2 \, d\hat{E}_{x,x} \xrightarrow{m \to \infty} 0,$$

as $f \in L^2(\hat{E}_{x,x})$ & $\mathbb{C} \setminus A_m \downarrow \emptyset$.

(6) Then $\|\Phi(f)x\|^2 = \lim_{n\to\infty} \|\Phi_0(f_n)x\|^2 \overset{V27(d)}{=} \lim_{n\to\infty} \int |f_n|^2 d\hat{E}_{x,x} =$

$$= \lim_{n\to\infty} \int_{A_n} |f|^2 d\hat{E}_{x,x} = \int |f|^2 d\hat{E}_{x,x}$$

(7) $\langle \Phi(f)x, y \rangle = \lim_{n\to\infty} \langle \Phi_0(f_n)x, y\rangle = \lim_{n\to\infty} \int f_n \, d\hat{E}_{x,y}$
$(x, y \in D(\Phi(f)))$

$$= \lim_{n\to\infty} \int_{A_n} f \, d\hat{E}_{x,y} = \int f \, d\hat{E}_{x,y} \qquad \text{(as } f \in L^1(|\hat{E}_{x,y}|) \text{ by (5) and } A_n \nearrow \mathbb{C} \text{)}$$

(8) $\Phi(f)$ unique:

Fix $x \in D(\Phi(f))$. If $z \in H$ is such that

$\langle \Phi(f)x, y\rangle = \langle z, y \rangle$, $y \in D(\Phi(f))$, then necessarily

$z = \Phi(f)x$ as $D(\Phi(f))$ is dense in H.