

## Proof of Theorem X.27

$E$  ... abstract spectral measure on  $H$

$\mathcal{A}$  ... the domain  $\sigma$ -algebra

$f: \mathcal{E} \rightarrow \mathbb{C}$  a bdd  $\mathcal{A}$ -measurable function

$$(1) \quad B(x, y) := \int f dE_{x, y}, \quad x, y \in H$$

•  $B(x, y)$  well-defined = ( $f$  measurable, bdd,  $E_{x, y}$  finite complex measure)

•  $B$  is sesquilinear ( $x \mapsto B(x, y)$  linear,  $y \mapsto B(x, y)$  conj. linear)  
by  $L^2(\mathcal{E}, \nu)$

$$|B(x, y)| \leq \int |f| d|E_{x, y}| \leq \|f\|_{\infty} \cdot \|E_{x, y}\| \leq \|f\|_{\infty} \|x\| \|y\|$$

So, by Riesz-Milgram Lemma there is (a unique)  $T \in L(H)$

$$\text{s.t. } \langle T x, y \rangle = \int f dE_{x, y}, \quad x, y \in H$$

$$\text{Moreover, } \|T\| \leq \|f\|_{\infty}$$

$$\text{Denote } \Phi_0(f) := T$$

(2) If  $f, g$  are bdd  $\mathcal{A}$ -measurable,  $f = g$   $E$ -a.e.

$$\text{(i.e. } \{\lambda \in \mathbb{C}; f(\lambda) \neq g(\lambda)\} \in \mathcal{N} = \{A \in \mathcal{A}, E(A) = 0\})$$

$$\Rightarrow \Phi_0(f) = \Phi_0(g)$$

$$\Gamma f = g \text{ } E\text{-a.e.} \Rightarrow \forall x, y \in H \quad \int f dE_{x, y} = \int g dE_{x, y}$$

$$\text{so } \int f dE_{x, y} = \int g dE_{x, y}$$

$$\text{By (1) we see } \Phi_0(f) = \Phi_0(g)$$

Therefore,  $\Phi_0$  is a well-defined mapping  $L^{\infty}(\mathcal{E}) \rightarrow L(H)$

$$\|\Phi_0(f)\| \leq \|f\|, \quad f \in L^{\infty}(\mathcal{E})$$

③ Clearly,  $\Phi_0$  is linear

$$\textcircled{4} \Phi_0(f)^* = \Phi_0(\overline{f})$$

$$\Gamma \langle \Phi_0(f)^* x, x \rangle = \langle x, \Phi_0(f)x \rangle = \overline{\langle \Phi_0(f)x, x \rangle} =$$

$$= \overline{\int f dE_{x,x}} = \int \overline{f} dE_{x,x} = \langle \Phi_0(\overline{f})x, x \rangle \quad \text{for } x \in H$$

$$\textcircled{5} A \in \mathcal{A} \Rightarrow \Phi_0(\chi_A) = E(A)$$

$$\Gamma \langle \Phi_0(\chi_A)x, y \rangle = \int \chi_A dE_{x,y} = E_{x,y}(A) = \langle E(A)x, y \rangle \quad \downarrow$$

$$\textcircled{6} \Phi_0(f \cdot g) = \Phi_0(f) \Phi_0(g)$$

$$\text{a) } f = \chi_A, g = \chi_B \Rightarrow \Phi_0(fg) = \Phi_0(\chi_{A \cap B}) = E(A \cap B) = E(A)E(B) = \underbrace{\Phi_0(\chi_A)}_{(v)} \underbrace{\Phi_0(\chi_B)}_{(5)} = \Phi_0(f) \Phi_0(g)$$

b)  $f, g$  simple  $\mathcal{A}$ -measurable

$$\left. \begin{aligned} f, g \text{ simple} &\Rightarrow \left\{ \begin{aligned} &\sum g_j; \Phi_0(f+g) = \Phi_0(f) + \Phi_0(g) \\ &\sum g_j; \Phi_0(gf) = \Phi_0(g) \Phi_0(f) \end{aligned} \right\} \text{ is a linear space} \end{aligned} \right\} \text{---||---}$$

so, by a) we deduce the validity for simple functions

c)  $g$  simple,  $f$  general

Fix  $x, y \in H$ . Find  $(f_n)$  simple Borel measurable

s.t.  $\|f_n\|_\infty \leq \|f\|_\infty$  and

$$f_n \rightarrow f \quad |E_{x,y}(f_n)| \rightarrow |E_{x,y}(f)| \quad \text{a.e.}$$

Lebesgue dom. conv.

$$\begin{aligned} \text{Th 1} \quad \langle \Phi_0(f) \Phi_0(g)_{x,y} \rangle &= \int f dE_{\Phi_0(g)_{x,y}} = \\ &= \lim_{n \rightarrow \infty} \int f_n dE_{\Phi_0(g)_{x,y}} = \lim_{n \rightarrow \infty} \langle \Phi_0(f_n) \Phi_0(g)_{x,y} \rangle = \\ &\stackrel{\square}{=} \lim_{n \rightarrow \infty} \langle \Phi_0(fg)_{x,y} \rangle = \lim_{n \rightarrow \infty} \int fg dE_{x,y} = \int fg dE_{x,y} \\ &= \langle \Phi_0(fg)_{x,y} \rangle \end{aligned}$$

Lebesgue dom. conv.

(a)  $fg$  general. Fix  $x,y \in H$   
 Find  $(g_n)$  simple Borel measurable,  $\|g_n\|_{\infty} \leq \|g\|_{\infty}$   
 $g_n \rightarrow g$   $|E_{x,y} + E_{x, \Phi_0(f)^* y}|$  - a.e.

$$\begin{aligned} \text{Th 1} \quad \langle \Phi_0(f) \Phi_0(g)_{x,y} \rangle &= \langle \Phi_0(g)_{x, \Phi_0(f)^* y} \rangle = \\ &= \int g dE_{x, \Phi_0(f)^* y} = \lim_{n \rightarrow \infty} \int g_n dE_{x, \Phi_0(f)^* y} = \\ &= \lim_{n \rightarrow \infty} \langle \Phi_0(g_n)_{x, \Phi_0(f)^* y} \rangle = \lim_{n \rightarrow \infty} \langle \Phi_0(f) \Phi_0(g_n)_{x,y} \rangle \\ &\stackrel{\square}{=} \lim_{n \rightarrow \infty} \langle \Phi_0(fg_n)_{x,y} \rangle = \lim_{n \rightarrow \infty} \int fg_n dE_{x,y} = \int fg dE_{x,y} \\ &= \langle \Phi_0(fg)_{x,y} \rangle \end{aligned}$$

$$\textcircled{7} \quad \|\Phi_0(f)_x\|^2 = \langle \Phi_0(f)_x, \Phi_0(f)_x \rangle = \langle \Phi_0(f)^* \Phi_0(f)_{x,x} \rangle$$

$$\stackrel{\textcircled{4}, \textcircled{6}}{=} \langle \Phi_0(\bar{f} \cdot f)_{x,x} \rangle = \int |f|^2 dE_{x,x}$$

(This proves (d))

$$\textcircled{8} \quad \Phi_0 \text{ is one-to-one} : \Phi_0(f) = 0 \stackrel{\textcircled{7}}{\Leftrightarrow} \forall x \in H \int |f|^2 dE_{x,x} = 0$$

$$\Rightarrow \forall x \in H : f = 0 \text{ } E_{x,x} \text{ - a.e.} \Rightarrow f = 0 \text{ } E \text{ - a.e.}$$

$$\Rightarrow f = 0 \text{ in } L^{\infty}(E)$$

⑨ proof of (a): By ③, ④, ⑥  $\Phi_0$  is a  $*$ -homomorphism  
 By ⑦ it is even a  $*$ -isomorphism, so, it  
 is an isometry.

⑩  $\sigma(\Phi_0(f)) = \text{ess-rng}(f)$

It is easy to see that  $\sigma(f) = \text{ess-rng}(f)$  in  $L^\infty(E)$

• Set  $\mathcal{B} := \Phi_0(L^\infty(E))$ . Then  $\mathcal{B}$  is a  $C^*$ -subalgebra of  $L(H)$   
 containing  $\mathbb{1} = \Phi_0(1)$ . So, for each  $f \in L^\infty(E)$  we have

$$\sigma_{L(H)}(\Phi_0(f)) = \sigma_{\mathcal{B}}(\Phi_0(f)) = \sigma(f) = \text{ess-rng}(f)$$

⑪  $\Phi_0(f)$  is always normal, as  $\mathcal{B}$  is commutative

$\Phi_0(f)$  self-adjoint  $\Leftrightarrow f$  self-adjoint, i.e. real-valued

It. 7  $\Phi_0(f) \geq 0 \Leftrightarrow \Phi_0(f)$  self-adjoint and  $\sigma(\Phi_0(f)) \subset (0, \infty)$



$f \geq 0$  E.a.o. by properties  
 of  $L^\infty(E)$ .

$$(12) \quad f \in C^\infty(E), \quad g \in \mathcal{C}(\sigma(\Phi_0(f))) \Rightarrow$$

$$\Rightarrow \Phi_0(g \circ f) = \tilde{g}(\Phi_0(f))$$

$\Gamma \quad \sigma(\Phi_0(f)) = \text{ess-rng}(f)$  is a compact set in  $\mathbb{C}$

$$\text{Let } \mathcal{Y} = \{g \in \mathcal{C}(\sigma(\Phi_0(f))) ; \Phi_0(g \circ f) = \tilde{g}(\Phi_0(f))\}$$

The  $\bullet$   $g \mapsto \tilde{g}(\Phi_0(f))$  is a  $*$ -isomorphism

$g \mapsto \Phi_0(g \circ f)$  is a  $*$ -homomorphism

So,  $\mathcal{Y}$  is a closed  $*$ -subalgebra of  $\mathcal{C}(\sigma(\Phi_0(f)))$

Moreover  $1 \in \mathcal{Y}$ , as

$$\tilde{1}(\Phi_0(f)) = \underline{1}, \quad \Phi_0(1 \circ f) = \Phi_0(1) = \underline{1},$$

here  $\mathcal{Y}$  contains constants

Finally,  $\text{id} \in \mathcal{Y}$ , as

$$\tilde{\text{id}}(\Phi_0(f)) = \Phi_0(f) = \Phi_0(\text{id} \circ f)$$

So,  $\mathcal{Y}$  separates points of  $\sigma(\Phi_0(f))$

Stone-Weierstrass theorem shows that  $\mathcal{Y} = \mathcal{C}(\sigma(\Phi_0(f)))$ .