XI.

Proof of Theorem 34 X Banach space, $K \subset X$ weakly compact $\Rightarrow \overline{aco}\,K$ is weakly compact

Proof: Let $T : X^* \to C(K)$ be defined by $T(x^*) = x^*\restriction K$

Then (similarly as in the proof of Theorem 33 (i) $\Rightarrow$ (ii)):

① T is a bdd linear operator, $\|T\| \le \sup\{\|x\|,\ x \in K\}$

② T is $w^* \to \tau_p$ cts, so $T(B_{X^*})$ is τ_p-compact.
Since it is bdd, it is weakly compact, so
T is weakly compact.

③ It follows from Theorem 33 (i) $\Rightarrow$ (iii) that
T' is $w^* \to w$ cts

④ $x \in K \Rightarrow$ define $\delta_x \in C(K)^*$ by $\delta_x(f) = f(x)$.
[i.e., it is the Dirac measure supported at x]
Then $T'(\delta_x) = \varkappa(x)$
[$T'(\delta_x)(x^*) = \delta_x(Tx^*) = x^*(x) = \varkappa(x)(x^*)$]

⑤ $\overline{aco\{\delta_x,\ x \in K\}}^{w^*} = B_{C(K)^*}$

[bipolar theorem: $\overline{aco\{\delta_x,\ x \in K\}}^{w^*} = (\{\delta_x,\ x \in K\}_0)^0 =$
$= (B_{C(K)})^0 = B_{C(K)^*}$]

⑥ So, $T'(B_{C(K)^*}) = T'\left(\overline{aco\{\delta_x,\ x \in K\}}^{w^*}\right) =$

$= \overline{T'(aco\{\delta_x,\ x \in K\})}^{w} = \overline{aco(\varkappa(K))}^{w} = \overline{aco}(\varkappa(K))$

$\subset$ T' is w^*-w cts by ③

$\supset$ the LHS is weakly compact

It follows that $\overline{aco}(\varkappa(K))$ is weakly compact, thus $\overline{aco}(K)$ is weakly compact as well!