

Proof of Lemma 27: Let K be compact, $A \subset C(K)$

\mathcal{E}_p -relatively compact. Then A is rel. \mathcal{E}_p -compact

(1) $x \in K \Rightarrow \{f(x); f \in A\}$ is bdd in \mathbb{F}

[If not, there is $(f_n) \subset A$ with $|f_n(x)| \rightarrow \infty$

But then (f_n) has no \mathcal{E}_p -cluster point]

(2) $r(x) := \sup \{|f(x)|; f \in A\}$, $x \in K$

By (1) $r(x) < \infty$ $\forall x \in K$, so

$$A \subset \{f \in C(K); |f(x)| \leq r(x) \text{ for } x \in K\} \subset$$

$$\subset \{f \in \mathbb{F}^K; |f(x)| \leq r(x) \text{ for } x \in K\} \subset$$

$$= \prod_{x \in K} \{x \in \mathbb{F}; |x| \leq r(x)\}$$

The last set is compact in \mathbb{F}^K by Tychonoff theorem.

So, it is enough to prove that $\overline{A} \subset C(K)$

(3) Fix $f \in \overline{A} \subset \mathbb{F}^K$. Suppose that $f \notin C(K)$.

Fix $x \in K$ s.t. f is not continuous at x

Fix $\varepsilon > 0$ s.t. $\forall U \ni x$, a neighborhood $\exists y \in U: |f(y) - f(x)| > \varepsilon$

Let us construct by induction $y_n \in K$, U_n neighborhoods of x
and $f_n \in A$ for $n \in \mathbb{N}$ such that

$$[i] |f_n(x) - f(x)| < \frac{1}{n}, \quad n \in \mathbb{N}$$

$$[ii] |f_n(y_k) - f(y_k)| < \frac{1}{n} \quad \text{for } n \in \mathbb{N}, k < n$$

$$[iii] U_n := \{z \in K; |f_k(z) - f_k(x)| < \frac{1}{n} \text{ for } k \leq n\}$$

$$[iv] y_n \in U_n \text{ \& } |f(y_n) - f(x)| > \varepsilon$$

(Construction: Find f_1 s.t. [i] holds (as $f \in \overline{A} \subset \mathbb{F}^K$)

Given $f_1, \dots, f_n, y_1, \dots, y_{n-1}$ define U_n by [iii], find y_n
as in [iv] (U_n is a neighborhood of x); find f_n satisfying [i], [ii]
(as $f \in \overline{A} \subset \mathbb{F}^K$)

K compact $\Rightarrow \exists$ (open) $y \in K$, a cluster point of (y_n)

A real ctsy compact $\Rightarrow \exists g \in C(K)$, a τ_p -cluster point of (f_n)

By [iv] : $\forall k \in \mathbb{N} : f_n(y_k) \rightarrow f(y_k)$

Since $g(y_k)$ is a cluster point of $(f_n(y_k))_n$,
we deduce $f(y_k) = g(y_k), k \in \mathbb{N}$.

By [iv] and [iii] : $y_n \in U_n$, so $\forall k \in \mathbb{N} : f_k(y_n) \rightarrow f_k(x)$

Since f_n is cts and y is a cluster point of (y_n) , $f_k(y)$ is a cluster point of $(f_k(y_n))_n$.
Therefore, $f_k(y) = f_k(x), k \in \mathbb{N}$

By [i] : $f_k(x) \rightarrow f(x)$, so $f_k(y) \rightarrow f(x)$.

Since g is a τ_p -cluster point of f_k ,

$g(y)$ is a cluster point of $(f_k(y))$, thus $g(y) = f(x)$

Finally, as $f(y_k) = g(y_k)$, g is cts and y is a cluster point of (y_k) , we deduce that $g(y)$ is a cluster point of $(g(y_k))$

Therefore, $f(x)$ is a cluster point of $f(y_k)$

" $g(y)$ " $g(y_k)$

But by [iv] $|f(x) - f(y_k)| > \epsilon$ for $k \in \mathbb{N}$, a contradiction.

Proof of Theorem XI.28 K compact Hausdorff space

$A \subset C(K)$, $f \in C(K)$

$f \in \overline{A}^{\tau_p} \Rightarrow \exists CCA$ ctble $f \in \overline{A}^{\tau_p}$

① $f \in \overline{A}^{\tau_p} \Leftrightarrow \forall x_1, \dots, x_k \in K \quad \forall n \in \mathbb{N} \exists g \in A:$
 $|g(x_j) - f(x_j)| < \frac{1}{n} \quad j=1, \dots, k$

② For $\delta, n \in \mathbb{N}$ and $g \in C(K)$ define

$$U_{\delta, n}(g) = \left\{ (x_1, \dots, x_k) \in K^k, |g(x_j) - f(x_j)| < \frac{1}{n} \quad j=1, \dots, k \right\}$$

Then $U_{\delta, n}(g)$ is an open subset of K^k

and

$$f \in \overline{A}^{\tau_p} \Leftrightarrow \forall \delta, n \in \mathbb{N} : \bigcup_{g \in A} U_{\delta, n}(g) = K^k \quad (*)$$

[This is just a reformulation of the equivalence in ①]

③ The proof:

$$f \in \overline{A}^{\tau_p} \stackrel{(*)}{\Rightarrow} \bigcup_{g \in A} U_{\delta, n}(g) = K^k \quad \text{for any } \delta, n \in \mathbb{N}$$

Fix δ, n . Since K^k is compact, there is a finite set

$$F_{\delta, n} \subset A \text{ s.t. } \bigcup_{g \in F_{\delta, n}} U_{\delta, n}(g) = K^k$$

Set $C := \bigcup_{\delta, n \in \mathbb{N}} F_{\delta, n}$. Then $C \subset A$ is ctble and

$$\forall \delta, n \in \mathbb{N} : \bigcup_{g \in C} U_{\delta, n}(g) = K^k \stackrel{(*)}{\Rightarrow} f \in \overline{C}^{\tau_p}$$

Proposition XI.29 K compact Hausdorff, $A \subset C(K)$
 (A, τ_p) compact & separable $\Rightarrow (A, \tau_p)$ metrizable

Proof: ① If K is metrizable, then it is separable. Fix $D \subset K$ cts dense set. Then the topology $\tau_p(D)$ (see Example VI.2 (5)) is metrizable (see Theorem V.22).

On A $\tau_p(D)$ coincides with τ_p , since A is compact and $\tau_p(D)$ is a weaker Hausdorff topology.

Thus (A, τ_p) is metrizable

② K general: Since A is τ_p -separable, fix a cts dense set $\{f_n, n \in \mathbb{N}\}$

Define $\varphi: K \rightarrow \mathbb{R}^{\mathbb{N}}$ by $\varphi(x) = (f_n(x))_{n=1}^{\infty}, x \in K$

Then φ is cts, hence $\varphi(K) =: L$ is compact. Since $\mathbb{R}^{\mathbb{N}}$ is metrizable, L is metrizable

Define $\varphi^*: C(L) \rightarrow C(K)$ by setting $\varphi^*(f) = f \circ \varphi, f \in C(L)$

Then φ^* is a linear isometry of $C(L)$ into $C(K)$

The linearity is clear.

$$\|\varphi^*(f)\|_{\infty} = \|f \circ \varphi\|_{\infty} = \sup_{x \in K} |f(\varphi(x))| = \sup_{y \in \varphi(K)} |f(y)| = \|f\|_{\infty}$$

\uparrow
 $\varphi(K) = L$

Moreover, φ^* is $\tau_p - \tau_p$ homeomorphism:

continuity: $x \in K \Rightarrow \varphi^*(f)(x) = f(\varphi(x))$ and

$f \mapsto f(\varphi(x))$ is τ_p -cts

continuity of the inverse: $y \in L$... fix some $x \in K$ with $\varphi(x) = y$

Then $\forall g \in \varphi^*(C(L))$:

$$(\varphi^*)^{-1}(g)(y) = g(x), \text{ so cts}$$

τ_p -cts

Further, $\varphi^*(\mathcal{C}(L))$ is τ_p -closed in $\mathcal{C}(K)$, as

$$\varphi^*(\mathcal{C}(L)) = \{f \in \mathcal{C}(K); \forall x, y \in K: \varphi(x) = \varphi(y) \Rightarrow f(x) = f(y)\}$$

• clearly: the set on RHS is τ_p -closed

• clearly: " \subset " holds

• \supset : $f \in$ RHS. Define $g: L \rightarrow \mathbb{F}$ by

$$g(\varphi(x)) = f(x), \quad x \in K. \text{ It is clearly}$$

well defined. It remains to show

that g is continuous:

$$H \subset \mathbb{F} \text{ closed} \Rightarrow g^{-1}(H) = \varphi(f^{-1}(H))$$

$f^{-1}(H)$ is closed (by continuity of f),

hence compact. Thus $\varphi(f^{-1}(H))$ is compact,

hence closed.

Finally, $f_n \in \varphi^*(\mathcal{C}(L))$, as $f_n = \pi_n \circ \varphi$, where

$\pi_n: L \rightarrow \mathbb{F}$ is the projection onto n -th coord.

so, $A \subset \varphi^*(\mathcal{C}(L))$.

It follows that A is homeomorphic to a subset of $(\mathcal{C}(L), \tau_p)$

[namely to $(\varphi^*)^{-1}(A)$]. Thus A is metrizable

by $\textcircled{1}$

As a corollary we get Theorem 26:

(a) K compact Hausdorff $\Rightarrow (C(K), \tau_p)$ is angelic

$\Gamma A C(C(K), \tau_p)$ rel. ctsly compact $\Rightarrow \overline{A}^{\tau_p}$ is compact
by Lemma 27,
so (c) is satisfied.

$f \in \overline{A}^{\tau_p} \xrightarrow{\text{Thm 28}} \exists CCA \text{ ctsly } f \in \overline{C}^{\tau_p}$. The \overline{C}^{τ_p}
is compact separable, thus metrizable (by Prop. 29)

It follows that there is $(f_n) \subset C$ s.t. $f_n \xrightarrow{\tau_p} f$ \downarrow

(5) X Borel space $\Rightarrow (t, \mathcal{K})$ angelic

$\Gamma (t, \mathcal{K}) \subset C((B_{t^*}, \mathcal{K}^*), \tau_p)$ by Theorem 15 \downarrow