Proposition XI.22 $\quad X$ HLCS, $K \subset X$ compact convex set

(a) $\mu \in P(K) \Rightarrow \exists!\, x \in K \quad \forall f : K \to \mathbb{R}$ cts affine $f(x) = \int f \, d\mu$

[This x is called the barycenter of μ, denoted by $r(\mu)$]

① Uniqueness: $x \neq y \overset{H-B}{\Rightarrow} \exists f \in X^* \quad f(x) \neq f(y)$ and f is cts affine

② $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$, where $\alpha_i \geq 0$ $\sum \alpha_i = 1$.

Then $r(\mu) = \sum \alpha_i x_i$

$\left(\left[f \text{ cts affine} \Rightarrow \int f d\mu = \sum \alpha_i f(x_i) = f\left(\sum \alpha_i x_i\right) \right] \right)$

③ Finitely supported measures are w^*-dense in $P(K)$.

Finitely supported probabilities $= \mathrm{co}\{\delta_x, x \in K\}$

Suppose $\mu \in P(K) \setminus \overline{\mathrm{co}\{\delta_x, x \in K\}}^{w^*}$. By H-B separation then there is $f \in C(K)$ (the dual of $(M(K), w^*)$) s.t. $\underset{\int f d\mu}{\underset{\parallel}{f(\mu)}} > \sup\{\underset{\int f d\delta_x = f(x)}{\underset{\parallel}{f(\delta_x)}}, x \in K\} = \sup f(K) = \max f(K)$

But $\underset{\mu \geq 0}{\int f d\mu \leq} \int \max f(K) \, d\mu \underset{\mu \text{ is a probability}}{=} \max f(K)$ $\quad \lightning$

④ Let $\mu \in P(K)$. By ③ there is a net μ_α of finitely supported probabilities s.t. $\mu_\alpha \xrightarrow{w^*} \mu$

By ② we have $r(\mu_\alpha) \in K$. Let $x \in K$ be a cluster point of the net $(r(\mu_\alpha))$. Then $x = r(\mu)$

$\Leftarrow$ f cts affine

$$\int f\,d\mu = \lim_\alpha \int f\,d\mu_\alpha = \lim_\alpha f(r(\mu_\alpha)) = f(x)$$

$\mu_\alpha \xrightarrow{w^*} \mu$ — f cts, x is a cluster point of $(r(\mu_\alpha))$

(3) The mapping $\mu \mapsto r(\mu)$ is cts and affine

- Affine: $r(t\mu + (1-t)\nu) = \cancel{\ldots}$ $t\,r(\mu) + (1-t)\,r(\nu)$

f cts affine $\Rightarrow$

$$f(t\,r(\mu) + (1-t)\,r(\nu)) = t\,f(r(\mu)) + (1-t)\,f(r(\nu)) =$$
$$= t\int f\,d\mu + (1-t)\int f\,d\nu = \int f\,d(t\mu + (1-t)\nu)$$

- cts: Since K is compact, the original topology on K coincides with the weak topology $\sigma(X, X^*)$. So, it is enough to show that $\forall x^* \in X^*$ $\mu \mapsto x^*(r(\mu))$ is w^*-cts.

But $x^*(r(\mu)) = \int x^*\,d\mu$, which is w^*-cts as $x^*|_K \in C(K)$.

Theorem XI.23 X HLCS, $K \subset X$ convex compact
$\forall x \in K \; \exists \mu \in P(K) \quad \mu(\overline{\text{ext}\, K}) = 1, \quad x = r(\mu)$

Proof: Consider the mapping $r : P(K) \to K$ provided by Prop. 22. It is continuous.

We wish to show that
$$r(\{\mu \in P(K);\ \mu(\overline{\text{ext}\, K}) = 1\}) = K$$

By theorem 18 we know that the image is dense in K

Further
$\{\mu \in P(K);\ \mu(\overline{\text{ext}\, K}) = 1\}$ is a closed subset of $P(K)$

[$\mu(\overline{\text{ext}\, K}) < 1 \Rightarrow \exists\ F \subset K \setminus \overline{\text{ext}\, K}$ compact
s.t. $\mu(F) > 0$. Urysohn lemma yields
$f : K \to [0,1]$ cts, $f|_F = 1$, $f|_{\overline{\text{ext}\, K}} = 0$

Then $\int f \, d\mu > 0$ ... $\{\nu \in P(K);\ \int f \, d\nu > 0\}$ is
a w^*-open set containing μ and disjoint with
$\{\nu \in P(K);\ \mu(\overline{\text{ext}\, K}) = 1\}$]

Thus the image is compact. So, the image is whole K