

Lemma XI.17 Let  $X$  be a vector space,  $\emptyset \neq A \subset X$  convex

(a)  $x \in A \Rightarrow (x \in \text{ext } A \Leftrightarrow \{x\} \text{ is a face of } A)$

[clear from definitions]

(b)  $F_1 \subset A$  face of  $A$ ,  $F_2 \subset F_1$  face of  $F_1$   
 $\Rightarrow F_2$  is a face of  $A$

[clearly  $F_2$  is nonempty and convex

$x, y \in A$   $\frac{x+y}{2} \in F_2 \Rightarrow x, y \in F_1$  (as  $F_2 \subset F_1$  and  $F_1$  is a face of  $A$ )

Hence  $x, y \in F_2$  (as  $F_2$  is a face of  $F_1$ ) ]

(c)  $X$  HLCs,  $A$  compact,  $A$  contains at least two points  
 $\Rightarrow \exists F \subsetneq A$  a face, closed in  $A$

[  $x, y \in A$ ,  $x \neq y$ . By H-B  $\exists f \in X^*$   $f(x) \neq f(y)$

Since  $A$  is compact and  $f$  cts,  $f$  attains max on  $A$

Let  $F = \{a \in A; f(a) = \max f(A)\}$

Then  $\emptyset \neq F \subsetneq A$  (as  $f(x) \neq f(y)$ ,  $f$  is not constant on  $A$ )

•  $F$  is closed and convex

[as  $f$  is cts and linear]

•  $\frac{x+y}{2} \in F$ ,  $x, y \in A$

$\Rightarrow \max f(A) = f(\frac{x+y}{2}) = \frac{1}{2}(f(x) + f(y)) \leq$

$\leq \frac{1}{2}(\max f(A) + \max f(A)) = \max f(A)$

$\Rightarrow$  there are equalities, i.e.  $x, y \in F$

So,  $F$  is a closed face. ]

Theorem XI. 18 (Krein-Milman)

$X$  HLCG,  $K \subset X$  compact convex  $\Rightarrow K = \overline{\text{co}} \text{ext} K$

Pf: Suppose  $K \neq \emptyset$ . Denote by  $\mathcal{F}$  the family of closed faces in  $K$ . If  $\mathcal{R} \subset \mathcal{F}$  is linearly ordered by " $\subset$ ", then  $\bigcap \mathcal{R} \in \mathcal{F}$  [the intersection is compact nonempty, ~~the~~ convex, the second property of a face is clear]

Thus, by Zorn's lemma, there is a minimal  $F \in \mathcal{F}$ .

By 17 (b,c)  $F$  is a singleton, so by 17(a) it consists of an extreme point. Let  $x \in \text{ext} F$ . Then  $x \in \text{ext} K$  (by 17(b,a)). Thus  $\text{ext} K \neq \emptyset$ .

If  $K \setminus \overline{\text{co}} \text{ext} K \neq \emptyset$ , fix  $x \in K \setminus \overline{\text{co}} \text{ext} K$ . By

H-B separation theorem  $\exists f \in X^*$   $f(x) > \sup f(\text{ext} K)$

Let  $F = \{y \in K, f(y) = \max f(K)\}$ . Then  $F$  is a closed face (cf. the proof of 17(a)). By the first part, we know that  $\text{ext} F \neq \emptyset$ , thus there is  $y \in \text{ext} F$ .

Since  $F \cap \text{ext} K = \emptyset$ , we have  $y \notin \text{ext} K$  } a contradiction.  
By 17(b,a) we deduce  $y \in \text{ext} K$  }