Theorem XI.12 Let $X$ be a normed space. Then $(X^*, bw^*)^* = \overline{\varkappa(X)}$

Proof ① Suppose $X$ is complete, i.e. $X$ is a Banach space

given $(x_n) \subset X$, $x_n \to 0$, the set $A = \{x_n, n \in \mathbb{N}\} \cup \{0\}$ is norm-compact. Since $X$ is complete, $\overline{\mathrm{aco}}\, A$ is also norm compact. (see Corollary V.29)

So, it is also weakly compact

Since $\sup\limits_{x \in \overline{\mathrm{aco}}\, A} |x^*(x)| = \sup\limits_{n \in \mathbb{N}} |x^*(x_n)|$ for each

$x^* \in X^*$, we deduce that $\cancel{bw^*} \subset bw^* \subset \mu(X^*, X)$,

Thm 6

Thus $(X^*, bw^*)^* = \varkappa(X)$

② $X$ is not complete. Let $\hat{X} = \overline{\varkappa(X)}$ denote its completion.

Then $X^* = (\hat{X})^*$ (as $X$ is dense in $\hat{X}$)

Moreover, on $rB_{X^*}$ the topologies $\sigma(X^*, X)$ and $\sigma(X^*, \hat{X})$ coincide

$\Gamma(rB_{X^*}, \sigma(X^*, \hat{X}))$ is compact and $\sigma(X^*, X)$ is a weaker Hausdorff topology )

It follows that $bw^*(X^*, X) = bw^*(X^*, \hat{X})$.

Hence, by ① we deduce $(X^*, bw^*) = \overline{\varkappa(X)}$.

Corollary XI.13 $X$ Banach space, $A \subset X^*$ convex

$\Rightarrow$ ($A$ is $w^*$-closed $\Leftrightarrow$ $\forall r > 0$ $A \cap rB_{X^*}$ is $w^*$-closed)

Pf: By theorem 13 $bw^*$ is an admissible topology on $X^*$, so by the mazur theorem a convex set is $w^*$-closed iff it is $bw^*$-closed

Corollary XI.14 $X$ Banach space, $f \in (X^*)^\#$. Then

$f \in \mathcal{X}(X) \Leftrightarrow f \restriction B_{X^*}$ is $w^*$ cts

Pf: $\Rightarrow$ clear

$\Leftarrow$ $f \restriction B_{X^*}$ $w^*$-cts $\Rightarrow$ $\forall r > 0$ $f \restriction rB_{X^*}$ is $w^*$-cts

$\Rightarrow$ $f$ is $bw^*$-cts $\Rightarrow$ $f \in \mathcal{X}(X)$ (Thm 13)