Proposition XI.11 Let X be a normed space. Then the bw^*-topology on X^* coincides with the topology of uniform convergence on sequences in X which converge to zero (in norm).

Proof ① $U \subset X^*$ bw^*-open, $x^* \in X^* \Rightarrow x^* + U$ bw^*-open

Γ • $U \subset X^*$ bw^*-open $\Rightarrow$ $\forall A \subset X^*$ bdd $A \cap U$ is (rel.) w^*-open in A.

Γ A bdd $\Rightarrow \exists r > 0 : A \subset rB_{X^*}$
$U \cap rB_{X^*}$ is w^*-open in rB_{X^*}, hence (by the definition of the subspace topology) $U \cap A$ is w^*-open in A $\lrcorner$

• $U \subset X^*$ bw^*-open, $x^* \in X^*$, $r > 0 \Rightarrow$

$$(x^* + U) \cap rB_{X^*} = x^* + \underbrace{U \cap (-x^* + rB_{X^*})}_{w^*\text{-open in } -x^* + rB_{X^*}}$$

as $y^* \mapsto x^* + y^*$ is w^*-homeomorphism, $(x^* + U) \cap rB_{X^*}$ is w^*-open in rB_{X^*} $\lrcorner$

② Let $(x_n) \in X$ be a sequence, $x_n \xrightarrow{\|\cdot\|} 0$
$A = \{x_n, n \in \mathbb{N}\}$
$q_A(x^*) = \sup \{|x^*(x_n)|, n \in \mathbb{N}\}$

Then $\forall c > 0$ $\{x^* \in X^*; q_A(x) < c\}$ is bw^*-open

Γ $r > 0$ ... Fix n_0 s.t. $\forall n \geq n_0 : \|x_n\| < \frac{c}{2r}$
Hence for $x^* \in rB_{X^*}$ and $n \geq n_0$: $|x^*(x_n)| \leq \|x^*\| \|x_n\| < \frac{c}{2}$
Therefore
$\{x^* \in rB_{X^*}; q_A(x) < c\} = \{x^* \in rB_{X^*}; |x^*(x_n)| < c \text{ for } n < n_0\}$,
which is w^*-open in rB_{X^*} $\lrcorner$

(3) Let $U \subset X^*$ be bw^*-open, $n \in \mathbb{N}$, $A \subset X$ s.t. $A^\circ \cap nB_{X^*} \subset U$
$\Rightarrow \exists\ F \subset \frac{1}{n} B_X$ finite s.t. $(A \cup F)^\circ \cap (n+1) B_{X^*} \subset U$

$F \subset \frac{1}{n} B_X$ finite $\Rightarrow$ denote $H_F = (A \cup F)^\circ \cap (n+1) B_{X^*} \setminus U$

Then H_F is w^*-compact for any F

($(n+1) B_{X^*} \cap U$ is w^*-open in $(n+1) B_{X^*}$, hence $(n+1) B_{X^*} \setminus U$ is w^*-closed in $(n+1) B_{X^*}$, thus H_F is w^*-closed)

We claim that $H_F = \emptyset$ for some F. Suppose not.

Since $H_{F_1} \cap H_{F_2} = \left((A \cup F_1)^\circ \cap (n+1) B_{X^*} \setminus U \right) \cap \left((A \cup F_2)^\circ \cap \cdots \right)$

$= \left((A \cup F_1)^\circ \cap (A \cup F_2)^\circ \right) \cap (n+1) B_{X^*} \setminus U =$

$= (A \cup F_1 \cup F_2)^\circ \cap (n+1) B_{X^*} \setminus U = H_{F_1 \cup F_2}$,

we see that (H_F, F) has a finite intersection property.

Thus $\bigcap_F H_F \neq \emptyset$. Fix $x^* \in \bigcap_F H_F$.

Then $x^* \in (n+1) B_{X^*} \setminus U$.

Further $\forall\ F \subset \frac{1}{n} B_X$ finite $x^* \in F^\circ$, so in part. $\forall x \in \frac{1}{n} B_X$: $|x^*(x)| \leq 1$, so $\|x^*\| \leq n$

So, $x^* \in nB_{X^*}$. Simultaneously $x^* \in A^\circ$, so

$x^* \in nB_{X^*} \cap A^\circ \subset U$, a contradiction

(4) $U \subset X^*$ bw^* open, $0 \in U \Rightarrow \exists\, A \subset X$, $A = \{t_n;\, n \in \mathbb{N}\}$ with $t_n \to 0$ s.t. $A^0 \subset U$

Γ • $U \cap B_{X^*}$ is rel. w^* open in B_{X^*} $\Rightarrow \exists\, F_1 \subset X$ finite s.t. $B_{X^*} \cap F_1^0 = \{x^* \in B_{X^*};\ |x^*(t)| \le 1 \text{ for } t \in F_1\} \subset U$

• Using (3) and induction we find for $n \in \mathbb{N}$ a finite set $F_{n+1} \subset \frac{1}{n} B_X$ s.t.

$$(F_1 \cup \dots \cup F_{n+1})^0 \cap (n+1) B_{X^*} \subset U$$

• Set $A = \bigcup_{n \in \mathbb{N}} F_n$. Then $A = \{t_n\}$, where $t_n \xrightarrow{\|\ \|} 0$

Moreover, $A^0 = \bigcup_{n \in \mathbb{N}} (A^0 \cap (n+1) B_{X^*}) \subset$

$$\subset \bigcup_{n \in \mathbb{N}} \left((F_1 \cup \dots \cup F_{n+1})^0 \cap (n+1) B_{X^*} \right) \subset U$$ $\lrcorner$

(5) Conclusion: Let $\mathcal{T}$ be the topology of uniform convergence on sequences converging to zero.

By (2) and (1) we see $\mathcal{T} \subset bw^*$

By (4): Any bw^*-neighborhood of 0 is a $\mathcal{T}$-nbhd of 0 using (1) we see $bw^* \subset \mathcal{T}$.