FUNCTIONAL ANALYSIS 1

WINTER SEMESTER 2021/2022

PROBLEMS TO CHAPTER II

Problems to Section II.1 – general weak topologies

Problem 1. Let $X = C([0,1])$ be equipped with the topology of pointwise convergence on $[0,1]$. Describe all the continuous linear functionals on X (i.e., describe X^*).

`Hint:` *Use Theorem II.4.*

Problem 2. Let X be a normed linear space which is not complete and let Y be its completion.

(1) Show that $X^* = Y^*$ and explain what this equality means.
(2) Show that the topologies $\sigma(Y^*, Y)$ and $\sigma(Y^*, X)$ are different.

`Hint:` *(2) Use Theorem II.4.*

Problem 3. Let X be a normed linear space, X^* its dual and X^{**} the second dual. Show that the weak and weak* topologies on X^* (i.e., the topologies $\sigma(X^*, X^{**})$ and $\sigma(X^*, X)$) coincide if and only if X is a reflexive Banach space.

`Hint:` *Use Theorem II.4.*

Problem 4. Let X be a normed linear space. Show that the canonical embedding $\varkappa : X \to X^{**}$ is a homeomorphism of (X, w) into (X^{**}, w^*).

`Hint:` *Use Proposition II.1(6).*

Problem 5. By Problem I.52 and the Introduction to functional analysis we know that $(\ell^p)^* = \ell^\infty$ for any $p \in (0,1]$.

(1) Show that the topologies $\sigma(\ell^\infty, \ell^p)$, $p \in (0,1]$, are pairwise distinct.
(2) Let $0 < p < q \le 1$. Which one of the topologies $\sigma(\ell^\infty, \ell^p)$ and $\sigma(\ell^\infty, \ell^q)$ is weaker?

`Hint:` *(1) Use Theorem II.4. (2) Use Proposition II.1(6).*

Problem 6. Let X be a vector space and let $M \subset X^\#$ separate points of X. Show that the topology $\sigma(X, M)$ is metrizable if and only if M does not contain an uncountable linearly independent subset.

`Hint:` *Use Theorem I.12 (or I.22) and Lemma II.3.*

Problems to Section II.2 – weak topologies on locally convex spaces

Problem 7. Let X be a normed linear space. Show that $(X, \|\cdot\|)$ is separable if and only if (X, w) is separable.

`Hint:` *Use Mazur theorem.*

Problem 8. Find an example of a Banach space X and a convex norm-closed subset of X^* which is not weak* closed.

`Hint:` *Consider for example $X = c_0$, hence $X^* = \ell^1$, and the closed convex hull of canonical unit vectors in ℓ^1. Another examples follow by Goldstine theorem.*

Problem 9. Let X and Y be LCS and let $T : X \to Y$ be a continuous linear mapping. Show that T is continuous as a mapping of (X, w) to (Y, w) as well.

`Hint:` *Use Proposition II.1(6).*

Problem 10. Let X and Y be LCS and let $T : X \to Y$ be a continuous linear mapping. For $\varphi \in Y^*$ define a mapping $T'\varphi : X \to \mathbb{F}$ by $T'\varphi = \varphi \circ T$.

(1) Show that $T'\varphi \in X^*$ for each $\varphi \in Y^*$.
(2) Show that the mapping $T' : \varphi \mapsto T'\varphi$ is a linear mapping of Y^* into X^*.
(3) Show that the mapping T' is continuous from (Y^*, w^*) to (X^*, w^*).

`Hint:` *(3) Use Proposition II.1(6).*

Problem 11. Let X and Y be normed linear spaces and let $T : Y^* \to X^*$ be a bounded linear operator. Show that there exists $S \in L(X, Y)$ such that $T = S'$, if and only if T is continuous as a mapping of (Y^*, w^*) into (X^*, w^*).

`Hint:` *Consider theoperator $T' : X^{**} \to Y^{**}$ and show that $T'(\varkappa(X)) \subset \varkappa(Y)$ using Corollary II.5(c).*

Problem 12. Let X be a Hilbert space and let (e_n) be an orthonormal sequence in X. Show that the sequence (e_n) converges weakly to zero.

`Hint:` *Use the representation of the dual to a Hilbert space and the Bessel inequality.*

Problem 13. Let X be a Hilbert space and let $(e_\gamma)_{\gamma\in\Gamma}$ be an orthonormal system in X. Show that the set $\{e_\gamma; \gamma \in \Gamma\} \cup \{\boldsymbol{o}\}$ is weakly compact.

`Hint:` *Using the representation of the dual to a Hilbert space and the Bessel inequality show that any weak neighborhood of zero contains all the elements of the orthonormal system except for finitely many.*

Problem 14. Let $X = c_0(\Gamma)$ or $X = \ell^p(\Gamma)$, where Γ is a set. Show that the set $\{\boldsymbol{o}\} \cup \{\boldsymbol{e}_\gamma; \gamma \in \Gamma\}$ is weakly compact ($\boldsymbol{e}_\gamma$ denotes the respective canonical unit vector).

`Hint:` *Using the representation of X^* show that any weak neighborhood of zero contains all the canonical unit vectors except for finitely many.*

Problem 15. Let $X = C([0, 1])$. Consider three topologies of X – the norm topology (i.e., the topology generated by the supremum norm), the weak one (i.e., the weak topology of the space $(X, \|\cdot\|_\infty)$ – let us denote it by w) and the topology of pointwise convergence on $[0, 1]$ (denote it by τ_p).

(1) Find a sequence (f_n) in X converging to zero in τ_p, which is not bounded in the norm.
(2) Show that there exists a τ_p-bounded set which is not norm-bounded.
(3) Let (f_n) be a norm-bounded sequence in X and let $f \in X$. Show that $f_n \xrightarrow{w} f$ if and only if $f_n \xrightarrow{\tau_p} f$.
(4) Does the equivalence in (3) hold without the assumption of norm boundedness?

`Hint:` *(2) Use the sequence from (1). (3) Use Riesz theorem on the representation of $C([0,1])^*$ and Lebesgue dominated convergence theorem. (4) Consider the sequence from (1).*

Problem 16. Show that in the space ℓ^1 weak and norm convergences of sequences coincide (i.e., ℓ^1 enjoys the **Schur property**).

`Hint:` *Proceed by contradiction: If not, then in ℓ^1 there exists a sequence $(\boldsymbol{x}_k)$ weakly converging to zero and a number $c > 0$ such that $\|\boldsymbol{x}_k\| > c$ for each $k \in \mathbb{N}$. Since $(\boldsymbol{x}_k)$ is bounded, without loss of generality $\|x_k\| = 1$ for each k. Weak convergence implies the convergence on each coordinate. By induction construct increasing sequences of natural numbers (k_j) and (m_j) such that $\sum_{l=m_j+1}^{m_{j+1}} \left|x_{k_j}(l)\right| > \frac{3}{4}$. Further find $\varphi \in \ell^\infty = (\ell^1)^*$ such that $\left|\varphi(x_{k_j})\right| > \frac{1}{2}$ for each j and deduce a contradiction.*

Problem 17. Show that the spaces c_0, ℓ^p for $p \in (1,\infty]$ and $\mathcal{C}([0,1])$ fail the Schur property.

`Hint:` *In any of these spaces find a sequence on the unit sphere weakly converging to zero. For $\mathcal{C}([0,1])$ use the description from Problem 15(3).*

Problem 18. Show that an infinitedimensional Hilbert space fails the Schur property.

`Hint:` *Use Problem 12.*

Problem 19. Show that the space $L^1([0,1])$ fails the Schur property.

`Hint:` *Let $T : L^2([0,1]) \to L^1([0,1])$ be the identity. Consider the ON basis (f_n) of the space $L^2([0,1])$ known from the theory of Fourier series and consider the sequence (Tf_n).*

Problem 20. Let X be normed linear space of infinite dimension.

(1) Show that any weak neighborhood of zero contains a nontrivial vector subspace of X.
(2) Show that S_X is a weakly dense subset of B_X.

`Hint:` *(1) Show that any weak neighborhood of zero contains the interesection of kernels of a finite number of functionals, and that this is a nontrivial vector subspace. (2) Use (1).*

Problem 21. Let X je normed linear space of infinite dimension.

(1) Show that any weak* neighborhood of zero in X^* contains a nontrivial vector subspace of X^*.
(2) Show that S_{X^*} is weak* dense subset of B_{X^*}.

`Hint:` *X^* has infinite dimension as well and the weak* topology is weaker than the weak one, hence one can apply Problem 20.*

Problem 22. Let X be a normed linear space. Show that the following assertions are equivalent:

(i) $\dim X < \infty$.
(ii) The weak and norm topologies on X coincide.
(iii) The weak* and norm topologies on X^* coincide.

Problems to Section II.3 – polars and their applications

Problem 23. Let X be a separable normed linear space. Show that (X^{**}, w^*) is separable.

`Hint:` *Use Goldstine theorem.*

Problem 24. Show that $((\ell^\infty)^*, w^*)$ is separable.

`Hint:` $\ell^\infty = (\ell^1)^*$.

Problem 25. Let X be a metrizable LCS. Show that (X^*, w^*) is σ-compact (i.e., it is the union of countably many compact subsets).

`Hint:` *Use Theorem II.14 and a countable base of neighborhoods of zero in X.*

Problem 26. Let X be a non-complete normed linear space and let Y be its completion. By Problem 2 we know that $X^* = Y^*$ and $\sigma(Y^*, X) \neq \sigma(X^*, X)$. Show that on the unit ball B_{X^*} the topologies $\sigma(Y^*, X)$ and $\sigma(X^*, X)$ coincide.

`Hint:` *By Corollary II.16 we know that $(B_{Y^*}, \sigma(Y^*, Y))$ is compact and the topology $\sigma(Y^*, X)$ is a weaker Hausdorff topology.*

Problem 27. Consider the space ℓ^∞ as the dual to ℓ^1. Show that on the unit ball of ℓ^∞ the weak* topology $\sigma(\ell^\infty, \ell^1)$ coincides with the topology of pointwise convergence (i.e. with the topology generated by the seminorms $\boldsymbol{x} = (x_k)_{k=1}^\infty \mapsto |x_n|$, $n \in \mathbb{N}$.

`Hint:` *Use Problem 26.*

Problem 28. Consider the space ℓ^1 as the dual to c_0. Show that on the unit ball of ℓ^1 the weak* topology $\sigma(\ell^1, c_0)$ coincides with the topology of pointwise convergence.

`Hint:` *Use Problem 26.*

Problem 29. Let $p \in (1, \infty)$. Show that on the unit ball of ℓ^p the weak topology coincides with the topology of pointwise convergence.

`Hint:` *Use Problem 26 and the reflexivity of ℓ^p.*

Problem 30. Show that on the unit ball of c_0 the weak topology coincides with the topology of pointwise convergence.

`Hint:` *Use Problems 4 and 27.*

Problem 31. Let X be a LCS and let X^* be its dual. For a nonempty $A \subset X^*$ define

$$q_A(x) = \sup\{|f(x)|\,;\, f \in A\}, \quad x \in X.$$

(1) Show that A is $\sigma(X^*, X)$-bounded if and only if $q_A(x) < \infty$ for each $x \in X$.
(2) Let A be $\sigma(X^*, X)$-bounded. Show that q_A is a seminorm on X.
(3) Must q_A be continuous on X?
(4) Let U be an absolutely convex neighborhood of zero in X. Show that $p_U = q_{U^\circ}$ (where p_U is the Minkowski functional).

`Hint:` *(3) Take an infinite-dimensional Banach space X equipped with the weak topology and $A = B_{X^*}$. (4) Use the bipolar theorem.*

Problem 32. Let X be a normed linear space, $C > 0$ and $f, g \in S_{X^*}$. Let $\|f|_{\ker g}\| \leq C$. Show that there exists $\alpha \in \mathbb{F}$, $|\alpha| = 1$ such that $\|f - \alpha g\| \leq 2C$.

`Hint:` *If $C \geq 1$ the statement is trivial, so suppose $C < 1$. By the Hahn-Banach theorem there exists $\tilde{f} \in X^*$, such that $\left\|\tilde{f}\right\| \leq C$ and $\tilde{f} = f$ on $\ker g$. Since $\ker g \subset \ker(f - \tilde{f})$, there is $\beta \in \mathbb{F}$ such that $f - \tilde{f} = \beta g$. Show that one can take $\alpha = \frac{\beta}{|\beta|}$.*

Problem 33. Let X be a Banach space. Let $f : X^* \to \mathbb{F}$ be a linear functional such that $f|_{B_{X^*}}$ is a weak* continuous mapping. Show that $f \in \varkappa(X)$.

`Hint:` *Since $f(B_{X^*})$ is a compact subset of $\mathbb{F}$, one gets $f \in X^{**}$. The case $f = 0$ is trivial, so without loss of generality $\|f\| = 1$. For $\varepsilon \in (0, 1)$ set $A_\varepsilon = \{x^* \in B_{X^*}; \operatorname{Re} f(x^*) \geq \varepsilon\}$ and $B_\varepsilon = \{x^* \in B_{X^*}; \operatorname{Re} f(x^*) \leq -\varepsilon\}$. Then A_ε and B_ε are nonempty disjoint weak* compact convex sets, hence by the separation theorem there exists $g \in \varkappa(X)$ such that $\sup \operatorname{Re} g(B_\varepsilon) < \inf \operatorname{Re} g(A_\varepsilon)$. Deduce that $\|f|_{\ker g}\| \leq \varepsilon$. Using Problem 32 then show that f belongs to the norm closure of $\kappa(X)$, so to $\kappa(X)$.*

Problem 34. Is the statement of the previous problem valid also for non-complete spaces?

`Hint:` *Use Problem 26.*