III. Elements of vector integration

Convention: In this chapter we will use the following notation:

- $(M, \mathcal{A})$ is a fixed measurable space, i.e., M is a nonempty set and $\mathcal{A}$ is a σ-algebra of subsets of M.
- (Ω, Σ, μ) is a fixed complete measure space, i.e., Ω is a nonempty set, Σ is a σ-algebra of subsets of Ω and μ is a non-negative σ-additive measure on Σ, which is moreover complete.
- X is a fixed Banach space over $\mathbb{F}$.

Remarks:

(1) (Ω, Σ) is a special case of a measurable space. Therefore, whatever is stated below for $(M, \mathcal{A})$, can be applied to (Ω, Σ) as well.
(2) We do not a priori assume that μ is finite or σ-finite, even though these cases are the most important ones.

III.1 Measurability of vector-valued functions

Definition. Let $f : M \to X$ be a mapping. The function f is said to be

- **simple**, if its range is a finite set, i.e., if $f = \sum_{j=1}^{k} x_j \chi_{A_j}$, where $x_1, \dots, x_k \in X$ and $A_1, \dots, A_k$ are nonempty pairwise disjoint subsets of M;
- **simple measurable**, if it can be expressed as above and, moreover, $A_1, \dots, A_k \in \mathcal{A}$;
- **(strongly) $\mathcal{A}$-measurable** if there exists a sequence (u_n) of simple measurable functions pointwise converging to f (i.e., such that $\lim\limits_{n\to\infty} \|u_n(t) - f(t)\| = 0$ for each $t \in M$);
- **Borel $\mathcal{A}$-measurable** if $f^{-1}(U) \in \mathcal{A}$ for each $U \subset X$ open;
- **weakly $\mathcal{A}$-measurable**, if $\varphi \circ f : M \to \mathbb{F}$ is (Borel) $\mathcal{A}$-measurable for each $\varphi \in X^*$.

Proposition 1.

(a) *Simple functions, simple measurable functions, strongly $\mathcal{A}$-measurable functions and weakly $\mathcal{A}$-measurable functions form vector spaces.*
(b) *Let (f_n) be a sequence of functions $f_n : M \to X$ pointwise converging to a function $f : M \to X$. If all the functions f_n are Borel $\mathcal{A}$-measurable (or weakly $\mathcal{A}$-measurable), the same holds for f.*
(c) *Let $f : M \to X$ be a function. Then*

$$f \text{ strongly } \mathcal{A}\text{-measurable} \Rightarrow f \text{ Borel } \mathcal{A}\text{-measurable} \Rightarrow f \text{ weakly } \mathcal{A}\text{-measurable}.$$

For simple functions all the mentioned types of measurability coincide.
(d) *If $f : M \to X$ is strongly $\mathcal{A}$-measurable, then $f(M)$ is a separable subset of X.*
(e) *If $f : M \to X$ is Borel $\mathcal{A}$-measurable, then $t \mapsto \|f(t)\|$ is an $\mathcal{A}$-measurable (scalar-valued) function.*

Remarks:

(1) Borel $\mathcal{A}$-measurable functions form a vector space if X is separable (by Theorem 3 below), in general they need not form a vector space.
(2) The converse implications in (c) fail, see Examples 6 below.

Lemma 2. *Let (f_n) be a sequence of strongly $\mathcal{A}$-measurable functions $f_n : M \to X$ pointwise converging to a function $f : M \to X$. Then f is strongly $\mathcal{A}$-measurable as well.*

Theorem 3 (Pettis). *Let $f : M \to X$ be a function. The following assertions are equivalent:*

(i) *f is strongly $\mathcal{A}$-measurable.*
(ii) *f is Borel $\mathcal{A}$-measurable and $f(M)$ is a separable subset of X.*
(iii) *f is weakly $\mathcal{A}$-measurable a $f(M)$ is a separable subset of X.*

Definition. Let $f : \Omega \to X$ be a mapping. The function f is said to be

- **(strongly) μ-measurable** if there exists a sequence (u_n) of simple measurable functions $u_n : \Omega \to X$ almost everywhere converging to f (i.e. such that $\lim_{n\to\infty} \|u_n(\omega) - f(\omega)\| = 0$ for almost all $\omega \in \Omega$);
- **Borel μ-measurable** (or **weakly μ-measurable**), it it is Borel Σ-measurable (or weakly Σ-measurable).

Remarks:

(1) Let $f : \Omega \to X$ be a function. Then:

$$f \text{ strongly } \mu\text{-measurable } \Rightarrow f \text{ Borel } \mu\text{-measurable } \Rightarrow f \text{ weakly } \mu\text{-measurable}$$

(2) If $f : \Omega \to X$ is (strongly) μ-measurable, then

$$\exists Y \subset\subset X \text{ separable } \exists N \in \Sigma : \mu(N) = 0 \ \&\ f(\Omega \setminus N) \subset Y.$$

A function satisfying this condition is called **essentially separably valued**.

Lemma 4. *Let (f_n) be a sequence of strongly μ-measurable functions $f_n : M \to X$ almost everywhere converging to a function $f : M \to X$. Then f is strongly μ-measurable as well.*

Theorem 5 (Pettis). *Let $f : \Omega \to X$ be a function. The following assertions are equivalent:*

(i) *f is strongly μ-measurable.*
(ii) *f is Borel μ-measurable and essentially separably valued.*
(iii) *f is weakly μ-measurable and essentially separably valued.*

Examples 6.

(1) *Let $\Omega = [0,1]$, let μ be the Lebesgue measure on $[0,1]$ and let Σ be the σ-algebra of all the Lebesgue measurable subsets of $[0,1]$. Consider the function $f : [0,1] \to \ell^2([0,1])$ defined by $f(t) = \boldsymbol{e}_t$, $t \in [0,1]$, where $\boldsymbol{e}_t$ denotes the respective canonical unit vector.*

Then f is weakly μ-measurable, but fails to be essentially separably valued, hence it is not strongly μ-measurable. It is neither Borel μ-measurable.

(2) *Let (Ω, Σ, μ) and f be as in (1). Let moreover $h : [0,1] \to [0,\infty)$ be any function. Then the function $h \cdot f$ is weakly μ-measurable as well. Furher, for $t \in [0,1]$ one has $\|h(t)f(t)\| = h(t)$. Therefore, if we choose h to be non-measurable, then $g = h \cdot f$ is weakly μ-measurable, but the function $t \mapsto \|g(t)\|$ is not measurable.*

(3) *Let $\Omega = [0,1]$, let Σ be the σ-algebra of all the subsets of $[0,1]$, let μ be the counting measure and let f be as in (1). Then f is Borel μ-measurable, but fails to be essentially separably valued, thus it is not strongly μ-measurable.*

Remark: The question, whether for a finite measure μ any Borel μ-measurable function is essentially separably valued (and hence strongly μ-measurable), is more complicated. The answer depends on additional axioms of the set theory.