II.2 Weak topologies on locally convex spaces

Theorem 6 (Mazur theorem). *Let X be a LCS and let $A \subset X$ be a convex set. Then:*

(a) $\overline{A}^w = \overline{A}$.

(b) *A is closed if and only if it is weakly closed.*

Corollary 7. *Let X be a metrizable LCS and let (x_n) be a sequence in X weakly converging to a point $x \in X$. Then there is a sequence (y_n) in X such that*

- $y_n \in \operatorname{co}\{x_k; k \geq n\}$ *for each* $n \in \mathbb{N}$*;*
- $y_n \to x$ *in (the original topology of)* X*.*

Theorem 8 (boundedness and weak boundedness). *Let X be a LCS and let $A \subset X$. Then A is bounded in X if and only if it is bounded in $\sigma(X, X^*)$.*

Proposition 9 (weak topology on a subspace). *Let X be a LCS and let $Y \subset\subset X$. Then the weak topology $\sigma(Y, Y^*)$ coincides with the restriction of the weak topology $\sigma(X, X^*)$ to Y.*

II.3 Polars and their applications

Definition. Let X be a LCS. Let $A \subset X$ and $B \subset X^*$ be nonempty sets. We define

$$A^{\triangleright} = \{f \in X^*; \forall x \in A : \operatorname{Re} f(x) \leq 1\}, \quad B_{\triangleright} = \{x \in X; \forall f \in B : \operatorname{Re} f(x) \leq 1\},$$
$$A^{\circ} = \{f \in X^*; \forall x \in A : |f(x)| \leq 1\}, \quad B_{\circ} = \{x \in X; \forall f \in B : |f(x)| \leq 1\},$$
$$A^{\perp} = \{f \in X^*; \forall x \in A : f(x) = 0\}, \quad B_{\perp} = \{x \in X; \forall f \in B : f(x) = 0\}.$$

The sets $A^{\triangleright}$ and $B_{\triangleright}$ are called **polars** of the sets A and B, the sets A° and $B_{\circ}$ are called **absolute polars** and the sets $A^{\perp}$ and $B_{\perp}$ are called **anihilators**.

Remarks:

(1) The terminology and notaion is not unified in the literature. Sometimes 'the polar' means 'the absolute polar', our polar is sometimes denoted by A°, $B_{\circ}$.

(2) If X is a Hilbert space and $A \subset X$, the symbol $A^{\perp}$ may have two different meanings – it may denote the above-defined anihilator or the orthogonal complement. It should be distinguished according to the context. However, these two possibilities are interrelated. Recall that in this case, given $x \in X$, the formula

$$f_x(y) = \langle y, x \rangle, \quad y \in X$$

defines a continuous linear functional on X and, moreover, $x \mapsto f_x$ is a (conjugate linear) isometry of X onto X^*. Then

$$\text{the anihilator of } A = \{f_x; x \in \text{ the orthogonal complement of } A\}.$$

(3) If X is Hausdorff and if we equip X^* by the weak* topology $\sigma(X^*, X)$, then $(X^*, w^*)^* = X$, and hence for any $B \subset X^*$ the (downward) polar $B_{\triangleright}$ by the previous definition coincide with the polar $B^{\triangleright}$ with respect to the space (X^*, w^*) and its dual X. Similarly for absolute polars and anihilators.

Example 10. *Let X be a normed linear space. Then*

(a) $(B_X)^{\triangleright} = (B_X)^{\circ} = B_{X^*}$,

(b) $(B_{X^*})_{\triangleright} = (B_{X^*})_{\circ} = B_X$.

Proposition 11 (polar calkulus). *Let X be a LCS and let $A \subset X$ be a nonempty set.*

(a) *The set $A^{\triangleright}$ is convex and contains the zero functional, A° is absolutely convex and $A^{\perp}$ is a subspace of X^*. All the three sets are moreover weak* closed.*
(b) $A^{\perp} \subset A^{\circ} \subset A^{\triangleright}$.
(c) *If A is balanced, then $A^{\triangleright} = A^{\circ}$. If $A \subset\subset X$, then $A^{\triangleright} = A^{\circ} = A^{\perp}$.*
(d) $\{\boldsymbol{o}\}^{\triangleright} = \{\boldsymbol{o}\}^{\circ} = \{\boldsymbol{o}\}^{\perp} = X^*$, $X^{\triangleright} = X^{\circ} = X^{\perp} = \{\boldsymbol{o}\}$.
(e) $(cA)^{\triangleright} = \frac{1}{c}A^{\triangleright}$ *and* $(cA)^{\circ} = \frac{1}{c}A^{\circ}$ *whenever* $c > 0$.
(f) *Let $(A_i)_{i\in I}$ be a nonempty family of nonempty subsets of X. Then $\left(\bigcup_{i\in I} A_i\right)^{\circ} = \bigcap_{i\in I} A_i^{\circ}$. The analogous formulas hold for polars and anihilators.*

Remark: Analogous statements hold for $B \subset X^*$ and for the sets $B_{\triangleright}$, $B_{\circ}$, $B_{\perp}$. There are just two differences: The sets $B_{\triangleright}$, $B_{\circ}$ and $B_{\perp}$ are weakly closed and for the validity of the second statement in (d) one needs to assume that X is Hausdorff.

Theorem 12 (bipolar theorem). *Let X be a LCS and let $A \subset X$ and $B \subset X^*$ be nonempty sets. Then*

$$\begin{aligned}
(A^{\triangleright})_{\triangleright} &= \overline{\text{co}}(A \cup \{\boldsymbol{o}\}) \ (= \overline{\text{co}}^{\sigma(X,X^*)}(A \cup \{\boldsymbol{o}\})), & (B_{\triangleright})^{\triangleright} &= \overline{\text{co}}^{\sigma(X^*,X)}(B \cup \{\boldsymbol{o}\}),\\
(A^{\circ})_{\circ} &= \overline{\text{aco}}A \ (= \overline{\text{aco}}^{\sigma(X,X^*)}A), & (B_{\circ})^{\circ} &= \overline{\text{aco}}^{\sigma(X^*,X)}B,\\
(A^{\perp})_{\perp} &= \overline{\text{span}}A \ (= \overline{\text{span}}^{\sigma(X,X^*)}A), & (B_{\perp})^{\perp} &= \overline{\text{span}}^{\sigma(X^*,X)}B.
\end{aligned}$$

Corollary 13. *Let X and Y be normed linear spaces and let $T \in L(X,Y)$. Then* $(\ker T)^{\perp} = \overline{T'(Y^*)}^{w^*}$.

Theorem 14 (Goldstine). *Let X be a normed linear space and let $\varkappa : X \to X^{**}$ be the canonical embedding. Then*

$$B_{X^{**}} = \overline{\varkappa(B_X)}^{\sigma(X^{**},X^*)}.$$

Theorem 15 (Banach-Alaoglu). *Let X be a LCS and let $U \subset X$ be a neighborhood of $\boldsymbol{o}$. Then:*

(a) *U° is a weak* compact subset of X^* (i.e., it is compact in the topology $\sigma(X^*,X)$).*
(b) *If X is moreover separable, U° is metrizable in the topology $\sigma(X^*,X)$.*

Corollary 16 (Banach-Alaoglu for normed spaces). *Let X be a normed linear space. Then (B_{X^*}, w^*) is compact. If X is separable, (B_{X^*}, w^*) is moreover metrizable.*

Corollary 17 (reflexivity and weak compactness). *Let X be a Banach space. Then X is reflexive if and only if B_X is weakly compact. If X is reflexive and separable, (B_X, w) is moreover metrizable.*

Corollary 18. *Let X be a reflexive Banach space. Then each bounded sequence in X admits a weakly convergent subsequence.*