II.1 General weak topologies and duality

Definition. Let X be a vector space over $\mathbb{F}$.

- By $X^{\#}$ we denote the **algebraic dual** of X, i.e., the vector space of all the linear functionals $f : X \to \mathbb{F}$.
- Let $M \subset X^{\#}$ be a nonempty set. By $\sigma(X, M)$ we denote the topology on X generated by the family of seminorms
$$\{x \mapsto |f(x)|; f \in M\}.$$
It is called **the weak topology generated by** M.

Proposition 1.

(1) *The space X is a LCS if it is equipped by the topology $\sigma(X, M)$.*
(2) *The topology $\sigma(X, M)$ is Hausdorff if and only if M* **separates points of** *X, i.e., if and only if for any $x \in X \setminus \{\boldsymbol{o}\}$ there exists $f \in M$ satisfying $f(x) \neq 0$.*
(3) *Functionals from M are continuous on $(X, \sigma(X, M))$.*
(4) *$\sigma(X, M)$ is the weakest (i.e., the smallest) topology on X in which all the functionals from M are continuous.*
(5) *$\sigma(X, M) = \sigma(X, \operatorname{span} M)$.*
(6) *Let T be a topological space and let $F : T \to X$ be any mapping. Then F is a continuous mapping of T to $(X, \sigma(X, M))$ if and only if $f \circ F$ is continuous on X for each $f \in M$.*

Examples 2.

(1) *Let X be a TVS. Then $X^* \subset\subset X^{\#}$, the topology $\sigma(X, X^*)$ is called* **the weak topology of** *X, sometimes it is denoted by w. If X is Hausdorff and locally convex, the topology $\sigma(X, X^*)$ is Hausdorff as well.*
(2) *Let X be a LCS (or, more generally, a TVS). Define a mapping $\varkappa : X \to (X^*)^{\#}$ by*
$$\varkappa(x)(f) = f(x), f \in X^*, x \in X.$$
Then $\varkappa(X)$ is a subspace of $(X^*)^{\#}$ separating points of X^*, hence the topology $\sigma(X^*, \varkappa(X))$ is Hausdorff. It is called **the weak* topology** *of X^*, it is denoted by $\sigma(X^*, X)$ or by w^*.*
(3) *Let Γ be a noenmpty set and let the space $\mathbb{F}^\Gamma$ be equipped by the product topology (cf. Example V.1(2)). The product topology equals $\sigma(\mathbb{F}^\Gamma, M)$ where $M = \{\boldsymbol{x} \mapsto \boldsymbol{x}(\gamma); \gamma \in \Gamma\}$.*
(4) *Let T be a topological space and let $\mathcal{C}(T, \mathbb{F})$ be the vector space of all the continuous functions on T. For $t \in T$ define the functional $\varepsilon_t \in \mathcal{C}(T, \mathbb{F})^{\#}$ by the formula*
$$\varepsilon_t(f) = f(t), \quad f \in \mathcal{C}(T, \mathbb{F}).$$
Then $M = \{\varepsilon_t; t \in T\}$ is a subset of $\mathcal{C}(T, \mathbb{F})^{\#}$ separating points of $\mathcal{C}(T, \mathbb{F})$, the topology $\sigma(\mathcal{C}(T, \mathbb{F}), M)$ is therefore Hausdorff. It is called **the topology of pointwise convergence**, *it is denoted by τ_p or by $\tau_p(T)$.*
(5) *Using the notation from the previous item, let moreover $D \subset T$ be a nonempty set and $M_D = \{\varepsilon_t; t \in D\}$. The topology $\sigma(\mathcal{C}(T, \mathbb{F}), M_D)$ is called* **the topology of pointwise convergence on** *D, it is denoted by $\tau_p(D)$. If D is dense in T, then the topology $\tau_p(D)$ is Hausdorff.*

Lemma 3. *Let X be a vector space and $f, f_1, \dots, f_k \in X^{\#}$. The following assertions are equivalent:*

(i) $f \in \operatorname{span}\{f_1, \dots, f_k\}$;
(ii) $\exists C > 0 \, \forall x \in X : |f(x)| \leq C \cdot \max\{|f_1(x)|, \dots, |f_k(x)|\}$;
(iii) $\bigcap_{j=1}^k \operatorname{Ker} f_j \subset \operatorname{Ker} f$.

Theorem 4. *Let X be a vector space and let $M \subset X^{\#}$ be a nonempty set. Then $(X, \sigma(X, M))^* = \operatorname{span} M$.*

Corollary 5.

(a) *Let X be a TVS and let $f \in X^{\#}$. Then f is continous on X (i.e., $f \in X^*$), if and only if it is weakly continous (i.e., $\sigma(X, X^*)$-continuous) on X.*
(b) *Let X be a TVS. Then $(X^*, \sigma(X^*, X))^* = \varkappa(X)$ (cf. Example 2(2)).*
(c) *Let X be a normed linear space and let $f \in X^{**}$. Then $f \in \varkappa(X)$ (where $\varkappa : X \to X^{**}$ is the canonical embedding), if and only if f is weak* continuous (i.e., $\sigma(X^*, X)$ continuous) on X^*.*