I.7 Separation theorems in locally convex spaces

Definition. Let X be a TVS over $\mathbb{F}$. By X^* we will denote the vector space of all the continuous linear functionals $f : X \to \mathbb{F}$. The space X^* is called **the dual space** (or **the dual**) of X.

Remarks:

(1) The dual of X is sometimes denoted by X'. The notation used in the literature is not unified. We will use for the 'continuous dual', i.e., for the space of *continuous* linear functionals, the symbol X^*. For the 'algebraic dual', i.e., the space of *all* linear functionals, we will use the symbol $X^\#$.

(2) We define X^* to be just a vector space, for the time being we do not equip it with any topology. Later (in the next chapter and in Functional Analysis 2) we will consider some natural topologies on X^*.

Theorem 33 (Hahn-Banach extension theorem). *Let X be a LCS over $\mathbb{F}$, $Y \subset\subset X$ and $f \in Y^*$. Then there exists $g \in X^*$ such that $g|_Y = f$.*

Remark: The assumption that X is locally convex is essential, the theorem fails in TVS.

Corollary 34 (separation from a subspace). *Let X be a LCS, Y a closed subspace of X and $x \in X \setminus Y$. Then there exists $f \in X^*$ such that $f|_Y = 0$ and $f(x) = 1$.*

Corollary 35 (a proof of density using Hahn-Banach theorem). *Let X be a LCS and let $Z \subset\subset Y \subset\subset X$. Then Z is dense in Y if and only if*

$$\forall f \in X^* : f|_Z = 0 \Rightarrow f|_Y = 0.$$

Theorem 36 (Hahn-Banach separation theorem). *Let X be a LCS, let $A, B \subset X$ be nonempty disjoint convex subsets.*

(a) *If the interior of A is nonempty, there exist $f \in X^* \setminus \{0\}$ and $c \in \mathbb{R}$ such that*

$$\forall a \in A\, \forall b \in B : \operatorname{Re} f(a) \le c \le \operatorname{Re} f(b).$$

(b) *If A is compact and B is closed, there exist $f \in X^*$ and $c, d \in \mathbb{R}$ such that*

$$\forall a \in A\, \forall b \in B : \operatorname{Re} f(a) \le c < d \le \operatorname{Re} f(b).$$

Corollary 37. *Let X be a LCS, let $A \subset X$ be a nonempty set and let $x \in X$. Then:*

(a) *$x \in X \setminus \overline{\operatorname{co}} A$ if and only if there exists $f \in X^*$ such that*

$$\operatorname{Re} f(x) > \sup\{\operatorname{Re} f(a); a \in A\}.$$

(b) *$x \in X \setminus \overline{\operatorname{aco}} A$ if and only if there exists $f \in X^*$ such that*

$$|f(x)| > \sup\{|f(a)|; a \in A\}.$$