IV.4 Ideals, complex homomorphisms and Gelfand transform

Definition. Let A be a Banach algebra. An **ideal** in A is a proper vector subspace $I \subset A$ such that $xy \in I$ and $yx \in I$ whenever $x \in I$ and $y \in A$. A **maximal ideal** in the algebra A is an ideal, which is maximal with respect to inclusion.

Remarks:

(1) Any ideal is a proper subalgebra. A proper subalgebra need not be an ideal.
(2) Also **left ideals** (defined by the implication $x \in I, y \in A \Rightarrow yx \in I$) and **right ideals** (defined similarly) are studied. Then an ideal is a subspace which is both a left ideal and a right ideal. We will not investigate unilateral ideals.

Proposition 18 (properties of ideals and of maximal ideals). *Let A be a unital Banach algebra.*

(a) *If I is an ideal in A, then $I \cap G(A) = \emptyset$.*
(b) *The closure of an ideal in A is again an ideal in A.*
(c) *Any ideal I in A is contained in a maximal ideal J.*
(d) *Any maximal ideal in A is closed.*

Examples 19.

(1) *If X is an infinite-dimensional Banach space, then $K(X)$ is a closed ideal in the Banach algebra $L(X)$.*
(2) *The only ideal in the matrix algebra M_n (where $n \in \mathbb{N}$) is the zero ideal.*
(3) *Let K be a compact Hausdorff space. Then all the closed ideals in the Banach algebra $\mathcal{C}(K)$ are the subspaces of the form*

$$\{f \in \mathcal{C}(K); f|_F = 0\}, \text{ where } F \subset K \text{ is a nonempty closed subset.}$$

Proposition 20 (factorization of an algebra). *Let A be a Banach algebra and let I be a closed ideal in A. Then the quotient Banach space A/I is a Banach algebra if the multiplication is defined by $q(x)q(y) = q(xy)$, where q is the quotient mapping of A onto A/I. Moreover, if A is commutative or unital, the same holds for A/I.*

Definition.

- Let A, B be Banach algebras. A mapping $h : A \to B$ is said to be a **homomorphism of Banach algebras** (shortly, a **homomorphism**), if it is linear and, moreover, $h(xy) = h(x)h(y)$ for $x, y \in A$.
- A **complex homomorphism** on a Banach algebra A is a homomorphism $h : A \to \mathbb{C}$.
- By $\Delta(A)$ we will denote the set of all the nonzero complex homomorphisms on A.

Remarks:

(1) In the definition of a homomorphism of Banach algebras there is no continuity requirement. In some important cases a homomorphism is automatically continuous (see, e.g., Proposition 21 or Proposition 31).
(2) If $h : A \to B$ is a homomorpism of Banach algebras, which is not identically zero, its kernel is an ideal in the algebra A.
(3) By the preceeding remark and Example 19(2) we see that for $n \geq 2$ one has $\Delta(M_n) = \emptyset$.
(4) The quotient mapping from Proposition 20 is a homomorphism of Banach algebras.

Proposition 21 (properties of complex homomorphisms). *Let A be a Banach algebra and let $h \in \Delta(A)$.*

- *If A has a unit e, then:*
  - *(a) $h(e) = 1$ and $\|h\| = 1$;*
  - *(b) $\ker h$ is a maximal ideal in A;*
  - *(c) $h(x) \neq 0$ for $x \in G(A)$.*
- *For a general Banach algebra A (unital or not) the following hold:*
  - *(d) There exists a unique $\tilde{h} \in \Delta(A^+)$ extending h (i.e., such that $\tilde{h}(x,0) = h(x)$ for $x \in A$);*
  - *(e) $\|h\| \leq 1$;*
  - *(f) $h(x) \in \sigma(x)$ for $x \in A$.*

Proposition 22 (properties of $\Delta(A)$). *Let A be a Banach algebra.*

(a) *If A is unital, then $\Delta(A)$ is a weak* compact subset of the unit sphere S_{A^*}.*

(b) *$\Delta(A^+) = \{\tilde{h}; h \in \Delta(A)\} \cup \{h_\infty\}$, where $\tilde{h}$ is the extension of h provided by Proposition 21(d) and $h_\infty(x,\lambda) = \lambda$ for $(x,\lambda) \in A^+$.*

(c) *If A has no unit, then $\Delta(A)$ is a subset of the unit ball B_{A^*} and $\Delta(A) \cup \{\boldsymbol{o}\}$ is weak* compact. Therefore, $\Delta(A)$ is locally compact in the weak* topology.*

Proposition 23 (complex homomorphisms and maximal ideals). *Let A be a unital Banach algebra.*

(1) *If I is an ideal in A of codimension one, there exists a unique $h \in \Delta(A)$ such that $I = \ker h$.*

(2) *If A is commutative, then $h \mapsto \ker h$ is a bijection of $\Delta(A)$ onto the set of all the maximal ideals in A.*

Definition. Let A be commutative Banach algebra.

- Let $x \in A$. For $h \in \Delta(A)$ we set $\hat{x}(h) = h(x)$. The function $\hat{x} : \Delta(A) \to \mathbb{C}$ is then called the **Gelfand transform of** x. It easily follows from definitions that $\hat{x}$ is a continuous complex function on $\Delta(A)$, moreover by Proposition 22(c) we see that $\hat{x} \in \mathcal{C}_0(\Delta(A))$.
- The **Gelfand transform of the algebra** A is the mapping $\Gamma : A \to \mathcal{C}_0(\Delta(A))$ defined by $\Gamma(x) = \hat{x}$, $x \in A$.

Theorem 24 (properties of the Gelfand transform). *Let A be a commutative Banach algebra and let $\Gamma : A \to \mathcal{C}_0(\Delta(A))$ be its Gelfand transform. Further, let $\Gamma^+ : A^+ \to \mathcal{C}(\Delta(A^+))$ be the Gelfand transform of the algebra A^+. To describe $\Delta(A^+)$ we use Proposition 22(b) (including the notation).*

(a) *Γ is a homomorphism of the algebra A into the algebra $C_0(\Delta(A))$.*

(b) *For $(x,\lambda) \in A^+$ one has*

$$\Gamma^+(x,\lambda)(\tilde{h}) = \Gamma(x)(h) + \lambda \quad \text{for } h \in \Delta(A),$$
$$\Gamma^+(x,\lambda)(h_\infty) = \lambda.$$

(c) *If A is unital, then*

$$\ker \Gamma = \operatorname{rad}(A) := \bigcap \{I : I \text{ is a maximal ideal in } A\}.$$

Hence, Γ is one-to-one (and so it is an isomorphism of the algebras A and $\Gamma(A) = \hat{A}$) if and only if $\operatorname{rad}(A) = \{0\}$ (i.e., if and only if A is **semisimple***).*

(d) *Γ is one-to-one if and only if Γ^+ is one-to-one.*

(e) *If A is unital, then for each $x \in A$ one has $\hat{x}(\Delta(A)) = \sigma(x)$.*

(f) *If A has no unit, then for each $x \in A$ one has $\sigma(x) = \hat{x}(\Delta(A)) \cup \{0\}$.*

(g) *$\|\hat{x}\| = r(x)$ for each $x \in A$.*

(h) *Γ is a continuous homomorphism, one has $\|\Gamma\| \leq 1$.*

(i) *Γ is a topological isomorphism of the algebras A and $\Gamma(A)$ if and only if it is one-to-one (see (c,d)) and $\hat{A} = \Gamma(A)$ is closed.*

(j) *$\Gamma(A)$ separates points of $\Delta(A)$.*