IV. Banach algebras and Gelfand transform

Convention: In this chapter all the Banach spaces are considered over the complex field (unless the converse is explicitly stated).

Remark: The real version of the theory of this chapter is studied as well, but it is quite different.

IV.1 Banach algebras – basic notions and properties

Definition.

- An **algebra** is a (complex) vector space A, equipped moreover with the operation of multiplication $\cdot$ which enjoys the following properties:
  - $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for $x, y, z \in A$;
  - $x \cdot (y + z) = x \cdot y + x \cdot z$ for $x, y, z \in A$;
  - $(x + y) \cdot z = x \cdot z + y \cdot z$ for $x, y, z \in A$;
  - $\alpha \cdot (x \cdot y) = (\alpha \cdot x) \cdot y = x \cdot (\alpha \cdot y)$ for $\alpha \in \mathbb{C}$ and $x, y \in A$.
- An algebra A is said to be **commutative**, if the multiplication is commutative, i.e., if
  - $x \cdot y = y \cdot x$ for $x, y \in A$.
- Let A be an algebra. An element $e \in A$ is said to be
  - a **left unit** if $e \cdot x = x$ for $x \in A$;
  - a **right unit** if $x \cdot e = x$ for $x \in A$;
  - a **unit** if $e \cdot x = x \cdot e = x$ for $x \in A$.

  An algebra admitting a unit is called **unital**.
- Let A be an algebra equipped moreover with a norm $\|\cdot\|$ satisfying
  - $\|x \cdot y\| \le \|x\| \cdot \|y\|$ for $x, y \in A$.

  Then A is said to be a **normed algebra.**
- A **Banach algebra** is a normed algebra A, which is complete in the metric generated by the norm.

Remarks:

(1) An algebra may have many left units or many right units.

(2) If an algebra has both a left unit and a right unit, they are equal. In particular, any algebra has at most one unit.

(3) If A is a nontrivial normed algebra with a unit e (nontrivial means $A \neq \{\boldsymbol{o}\}$), then $\|e\| \ge 1$.

Examples 1 (examples of Banach algebras).

(1) *The complex field is a unital commutative Banach algebra.*

(2) *Let K be a compact Hausdorff space. Then $\mathcal{C}(K)$, the space of all the complex-valued continuous functions on K equipped with the supremum norm and with the pointwise multiplication (i.e., $(f \cdot g)(x) = f(x) \cdot g(x)$ for $f, g \in \mathcal{C}(K)$ and $x \in K$) is a unital commutative Banach algebra. Its unit is the constant function equal to* 1.

(3) *Let T be a locally compact Hausdorff space which is not compact (e.g., $T = \mathbb{R}^n$). Let the space*

$$\mathcal{C}_0(T) = \{f : T \to \mathbb{C} \text{ continuous}; \forall \varepsilon > 0 : \{x \in T; |f(x)| \ge \varepsilon\} \text{ is a compact subset of } T\}$$

be equipped with the supremum norm and with the pointwise multiplication. Then $\mathcal{C}_0(T)$ is a commutative Banach algebra which has no unit.

(4) *For $n \in \mathbb{N}$ let M_n be the space of all the complex square matrices of order n, equipped with the matrix norm and with the matrix multiplication. Then M_n is a unital Banach algebra. Its unit is the unit matrix. If $n \ge 2$, M_n is not commutative.*

(5) *Let X be a Banach space and let $L(X)$ be the space of all the bounded linear operators on X equipped with the operator norm. If we define the multiplication on $L(X)$ as the composition of operators (i.e., $S \cdot T = S \circ T$ for $S, T \in L(X)$), then $L(X)$ is a unital Banach algebra. Its unit is the identity mapping. If $\dim X \geq 2$, the algebra $L(X)$ is not commutative.*

(6) *Let X be a Banach space and let $K(X)$ be the space of all the compact operators on X. Then $K(X)$ is a closed subalgebra of $L(X)$, hence it is a Banach algebra. The algebra $K(X)$ is unital if and only if X is finite-dimensional. $K(X)$ is commutative if and only if $\dim X = 1$.*

(7) *The Banach space $L^1(\mathbb{R}^n)$ becomes a commutative Banach algebra, if we define the multiplication as the convolution. This algebra has no unit.*

(8) *The Banach space $\ell^1(\mathbb{Z})$, equipped with the multiplication $*$ (called also convolution) defined by*

$$(x_n)_{n\in\mathbb{Z}} * (y_n)_{n\in\mathbb{Z}} = \left(\sum_{k\in\mathbb{Z}} x_k y_{n-k} \right)_{n\in\mathbb{Z}}, \quad (x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}),$$

is a unital commutative Banach algebra. It unit is the canonical vector $\boldsymbol{e}_0$.

(9) *Let μ be a normalized Lebesgue measure on $[0, 2\pi)$ (i.e., $\mu = \frac{1}{2\pi}\lambda$, where λ is a Lebesgue measure on $[0, 2\pi)$). Then the Banach space $L^1(\mu)$, equipped with the multiplication $*$ (called also convolution) defined by*

$$\begin{aligned} f * g(x) &= \int_{[0,2\pi)} f(y) g((x - y) \mod 2\pi) \,\mathrm{d}\mu(y) \\ &= \frac{1}{2\pi} \int_{[0,2\pi)} f(y) g((x - y) \mod 2\pi) \,\mathrm{d}y, \quad f, g \in L^1(\mu), x \in [0, 2\pi), \end{aligned}$$

is a commutative Banach algebra. This algebra has no unit.

Proposition 2 (adding a unit).

(a) *Let A be an algebra. Let A^+ denote the vector space $A \times \mathbb{C}$ equipped with the multiplication defined by*

$$(x, \lambda) \cdot (y, \mu) = (x \cdot y + \lambda y + \mu x, \lambda\mu), \quad (x, \lambda), (y, \mu) \in A^+.$$

Then A^+ is an algebra and the element $(\boldsymbol{o}, 1)$ is its unit. Moreover, $\{(a, 0); a \in A\}$ is a subalgebra of A^+, which is isomorphic to the algebra A.

(b) *If A is a Banach algebra, then A^+ is a unital Banach algebra, if we define the norm by $\|(x, \lambda)\| = \|x\| + |\lambda|$, $(x, \lambda) \in A^+$. Moreover, $\{(a, 0); a \in A\}$ is then a closed subalgebra of A^+, which is isometrically isomorphic to the Banach algebra A.*

Remarks:

(1) The algebraic structure of the algebra A^+ is uniquely determined, for the norm on A^+ it is not the case. The given norm is one of the possible ones, later we will see other possibilities, which are natural in some special cases.

(2) The procedure of adding a unit is important mainly in case A is not unital. However, it has a sense also in case A is unital. If A has a unit e, the unit of A^+ is $(\boldsymbol{o}, 1)$ and the element $(e, 0)$ is not a unit anymore. This element is the unit of the subalgebra $\{(a, 0), a \in A\}$.

Proposition 3 (renorming of a Banach algebra). *Let $(A, \|\cdot\|)$ be a nontrivial Banach algebra with a unit e. Then there exists an equivalent norm $|||\cdot|||$ on A such that $(A, |||\cdot|||)$ is also a Banach algebra and, moreover, $|||e||| = 1$.*

Convention: By a **unital Banach algebra** we will mean in the sequel a nontrivial Banach algebra, which has a unit and the unit has norm one.

Proposition 4. *Let A be a Banach algebra. Then:*

(a) $x \cdot \boldsymbol{o} = \boldsymbol{o} \cdot x = \boldsymbol{o}$ *for* $x \in A$.
(b) *The multiplication is continuous as a mapping of $A \times A$ to A.*

Definition. Let A be a Banach algebra with a unit e.

- The element $y \in A$ is said to be an **inverse element** (or just an **inverse**) of an element $x \in A$ if
$$x \cdot y = y \cdot x = e.$$
- An element $x \in A$ is called **invertible** if it admits an inverse.
- The set of all the invertible elements of A is denoted by $G(A)$.

Remark. Let A be a Banach algebra with a unit e and let $x \in A$. If $y \in A$ satisfies $x \cdot y = e$, it is called a **right inverse** of x; if it satisfies $y \cdot x = e$, it is called a **left inverse**. An element x can have many different right inverses, or many different left inverses. However, if x has both a right inverse and a left inverse, it is invertible. Its inverse is uniquely determined and it is simultaneuously the unique right inverse and the unique left inverse. The inverse of x is denoted by x^{-1}.

Proposition 5 (on multiplication of invertible elements). *Let A be a unital Banach algebra.*

(a) *Let $x, y \in G(A)$. Then $x \cdot y \in G(A)$ and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.*
(b) *$G(A)$ equipped with the operation of multiplication is a group.*
(c) *If the elements $x_1, \dots, x_n \in A$ commute (i.e., $x_j \cdot x_k = x_k \cdot x_j$ for $j, k \in \{1, \dots, n\}$), then $x_1 \cdots x_n \in G(A)$ if and only if $\{x_1, \dots, x_n\} \subset G(A)$.*

Lemma 6 (Neumann's series). *Let A be a Banach algebra with a unit e.*

(a) *Let $x \in A$ such that $\|x\| < 1$. Then $e - x \in G(A)$ and, moreover,*
$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n,$$
where the series converges absolutely.
(b) *If $x \in G(A)$, $h \in A$ and $\|h\| < \frac{1}{\|x^{-1}\|}$, then $x + h \in G(A)$ and, moreover,*
$$(x+h)^{-1} = x^{-1} \cdot \sum_{n=0}^{\infty} (-1)^n (h \cdot x^{-1})^n \qquad \text{and} \qquad \|(x+h)^{-1} - x^{-1}\| \leq \frac{\|x^{-1}\|^2 \|h\|}{1 - \|x^{-1}\| \|h\|}.$$

Theorem 7 (topological properties of the group of invertible elements). *Let A be a unital Banach algebra. Then*

(1) *$G(A)$ is an open subset of A,*
(2) *the mapping $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$ onto $G(A)$,*
(3) *if (x_n) is a sequence in $G(A)$ which converges in A to some $x \notin G(A)$, then $\|x_n^{-1}\| \to \infty$.*