

I.1 Linear topologies and their generating

Basic notation:

\mathbb{R} ... the field of real numbers

\mathbb{C} ... the field of complex numbers

\mathbb{F} ... the field \mathbb{R} or \mathbb{C}

If X is a vector space over \mathbb{F} , the zero vector is denoted by \mathbf{o} (and sometimes by 0).

If X is a vector space over \mathbb{F} , $Y \subset\subset X$ means that Y is a subspace of X .

Definition. A **topological vector space over \mathbb{F}** is a pair (X, \mathcal{T}) where X is a vector space over \mathbb{F} and \mathcal{T} is a topology on X with the following two properties:

- (1) The mapping $(x, y) \mapsto x + y$ is a continuous mapping of $X \times X$ into X .
- (2) The mapping $(t, x) \mapsto tx$ is a continuous mapping of $\mathbb{F} \times X$ into X .

The term *topological vector space* will be abbreviated by **TVS**. If (X, \mathcal{T}) is moreover Hausdorff, we write **HTVS**.

The symbol $\mathcal{T}(\mathbf{o})$ will denote the family of all the neighborhoods of \mathbf{o} in (X, \mathcal{T}) .

Definition. Let (X, \mathcal{T}) be a TVS. The space X is said to be **locally convex**, if there exists a base of neighborhoods of zero consisting of convex sets. The term *locally convex TVS* will be abbreviated by **LCS**, if it is moreover Hausdorff, then by **HLCS**.

Examples 1.

- (1) Let $(X, \|\cdot\|)$ is a normed linear space and let \mathcal{T} be the topology generated by the norm (i.e., generated by the metric induced by the norm). Then (X, \mathcal{T}) is HLCS.
- (2) Let Γ be any nonempty set. Then \mathbb{F}^Γ is HLCS, if it is equipped by the product topology.
- (3) The space $\mathcal{C}(\mathbb{R}, \mathbb{F})$ of continuous functions on \mathbb{R} is HLCS, if it is equipped by the topology of locally uniform convergence. This topology is generated, for example, by the metric

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, \max\{|f(x) - g(x)|; x \in [-n, n]\}\}, \quad f, g \in \mathcal{C}(\mathbb{R}, \mathbb{F}).$$

- (4) Let $\Omega \subset \mathbb{C}$ be an open set. Then the space $H(\Omega)$ of holomorphic functions on Ω is HLCS, if it is equipped by the topology of locally uniform convergence. This topology is generated, for example, by the metric

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, \max\{|f(z) - g(z)|; z \in K_n\}\}, \quad f, g \in H(\Omega),$$

where (K_n) is an exhausting sequence of compact subsets of Ω (i.e., sequence of compact subsets satisfying $K_n \subset \text{Int } K_{n+1}$ for each $n \in \mathbb{N}$ and $\bigcup_n K_n = \Omega$).

- (5) The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is HLCS, if it is equipped by the topology generated by the standard metric.
- (6) Let $\Omega \subset \mathbb{R}^d$ be an open set and $K \subset \Omega$ a compact subset. Then the space $\mathcal{D}_K(\Omega)$ is HLCS, if it is equipped by the topology generated by the standard metric.
- (7) Let (Ω, Σ, μ) be a measure space (where μ is a nonnegative measure) and $p \in (0, 1)$. Then the space $L^p(\Omega, \Sigma, \mu)$ consisting of equivalence classes of measurable functions $f : \Omega \rightarrow \mathbb{F}$ satisfying $\int_{\Omega} |f|^p d\mu < \infty$ is HTVS, if it is equipped by the topology generated by the metric

$$\rho(f, g) = \int_{\Omega} |f - g|^p d\mu, \quad f, g \in L^p(\Omega, \Sigma, \mu).$$

If, for example, $\Omega = [0, 1]$ and μ is the Lebesgue measure or $\Omega = \mathbb{N}$ and μ is the counting measure, then this space fails to be locally convex.

Remark: If (X, \mathcal{T}) is TVS, then \mathcal{T} is translation invariant. I.e., if $A \subset X$ and $x \in X$, then A is open if and only if $x + A$ is open. It follows that $A \subset X$ is a neighborhood of $x \in X$ if and only if $-x + A \in \mathcal{T}(\mathbf{o})$. Therefore the family $\mathcal{T}(\mathbf{o})$ uniquely determines the topology \mathcal{T} .

Definition. Let X be a vector space over \mathbb{F} and $A \subset X$. We say that the set A is

- **convex**, if $tx + (1 - t)y \in A$ whenever $x, y \in A$ and $t \in [0, 1]$;
- **symmetric**, if $A = -A$;
- **balanced**, if $\alpha A \subset A$ whenever $\alpha \in \mathbb{F}$ is such that $|\alpha| \leq 1$;
- **absolutely convex**, if it is convex and balanced;
- **absorbing**, if for each $x \in X$ there exists $t > 0$ such that $\{sx; s \in [0, t]\} \subset A$.

Definition. Let X be a vector space over \mathbb{F} and $A \subset X$. By the **convex hull** (**balanced hull**, **absolutely convex hull**) of A we mean the smallest convex (balanced, absolutely convex) set containing A . This set is denoted by $\text{co}(A)$ ($\text{b}(A)$, $\text{aco}(A)$, respectively).

Proposition 2. Let X be a vector space over \mathbb{F} and $A \subset X$.

- (a) If $\mathbb{F} = \mathbb{R}$, then A is absolutely convex, if and only if it is convex and symmetric.
- (b) $\text{co}(A) = \{t_1x_1 + \dots + t_kx_k; x_1, \dots, x_k \in A, t_1, \dots, t_k \geq 0, t_1 + \dots + t_k = 1\}$.
- (c) $\text{b}(A) = \{\alpha x; x \in A, \alpha \in \mathbb{F}, |\alpha| \leq 1\}$.
- (d) $\text{aco}(A) = \text{co}(\text{b}(A))$.

Proposition 3. Let (X, \mathcal{T}) be a TVS and $U \in \mathcal{T}(\mathfrak{o})$. Then:

- (i) U is absorbing.
- (ii) There exists $V \in \mathcal{T}(\mathfrak{o})$ such that $V + V \subset U$.
- (iii) There exists $V \in \mathcal{T}(\mathfrak{o})$ open and balanced such that $V \subset U$.
- (iv) If X is locally convex, there exists $V \in \mathcal{T}(\mathfrak{o})$ open and absolutely convex such that $V \subset U$.

Theorem 4.

- (1) Let (X, \mathcal{T}) be a TVS. Then there exists \mathcal{U} , a base of neighborhoods of \mathfrak{o} with the following properties:

- (i) The elements of \mathcal{U} are absorbing, open and balanced.
- (ii) For any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V + V \subset U$.

If X is locally convex, it is possible to choose \mathcal{U} such that its elements are moreover absolutely convex. If X is moreover Hausdorff, then $\bigcap \mathcal{U} = \{\mathfrak{o}\}$.

- (2) Conversely, let X be a vector space and \mathcal{U} a family of subsets of X with the following properties:
 - (i) The elements of \mathcal{U} are absorbing and balanced.
 - (ii) For any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V + V \subset U$.
 - (iii) For any $U, V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $W \subset U \cap V$.

Then there exists a unique topology \mathcal{T} on X such that (X, \mathcal{T}) is a TVS and \mathcal{U} is a base of neighborhoods of \mathfrak{o} . If all the elements of \mathcal{U} are moreover absolutely convex, \mathcal{T} is locally convex. Further, if $\bigcap \mathcal{U} = \{\mathfrak{o}\}$, then \mathcal{T} is Hausdorff.

Examples 5.

- (1) Let T be a Hausdorff topological space and let $\mathcal{C}(T, \mathbb{F})$ denote the space of all the continuous functions on T . Then the family

$$\mathcal{U} = \left\{ \{f \in \mathcal{C}(T, \mathbb{F}); \max_{x \in K} |f(x)| < c\}; K \subset T \text{ compact}, c > 0 \right\}$$

forms a base of neighborhoods of \mathfrak{o} in a topology, in which $\mathcal{C}(T, \mathbb{F})$ is HLCS. It is the topology of uniform convergence on compact subsets. If T is locally compact, it is the topology of locally uniform convergence.

- (2) Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\mathcal{D}(\Omega)$ denotes the space of all the test functions on Ω . Then the family

$\mathcal{U} = \{U \subset \mathcal{D}(\Omega) \text{ absolutely convex and absorbing};$

$$\forall K \subset \Omega \text{ compact} : U \cap \mathcal{D}_K(\Omega) \text{ is a neighborhood of zero in } \mathcal{D}_K(\Omega)\}$$

forms a base of neighborhood of \mathfrak{o} in a topology, in which $\mathcal{D}(\Omega)$ is HLCS.