$X = \{ [f] \,;\, f : \mathbb{R} \to \mathbb{F}$ measurable s.t. $\forall p \in (0,\infty) : [f] \in \mathcal{L}^p(\mathbb{R}) \}$

$$\mathcal{U} = \Big\{ \{ [f] \in X \,;\, \| [f] \|_{p_1} < \varepsilon, \dots, \| [f] \|_{p_k} < \varepsilon \} \,; \quad p_1, \dots, p_k \in (0,\infty),\ \varepsilon > 0 \Big\}$$

Recall: $\| [f] \|_p = \begin{cases} \int_0^\infty |f|^p & \text{if } p \in (0,1) \\ \left(\int_0^\infty |f|^p \right)^{1/p} & \text{if } p \in [1,\infty) \end{cases}$

(1) $\mathcal{U}$ is a base of nbhds of 0 in a Hausdorff ~~locally~~ linear topology

Γ check the axioms.

- $U \in \mathcal{U} \Rightarrow U$ is balanced

  Γ clear, as for $|\lambda| \le 1$ we have

  $$\| [\lambda f] \|_p = \begin{cases} |\lambda|^p \| f \|_p \le \| f \|_p & \text{if } p < 1 \\ |\lambda| \, \| f \|_p \le \| f \|_p & \text{if } p \ge 1 \end{cases} \lrcorner$$

- $U \in \mathcal{U} \Rightarrow U$ absorbing

  Γ $U = U_{p_1, \dots, p_k, \varepsilon}$ $\qquad f \in X$

  $\Rightarrow \| [f] \|_{p_j} < \infty$ for each j.

$t \geq 0$

$$\| t\,[f] \|_{p_j} = \begin{cases} t\, \|[f]\|_{p_j} & p_j \geq 1 \\ t^{p}\, \|[f]\|_{p_j} & p_j \leq 1 \end{cases}$$

hence, if $t \geq 0$ is small enough, all those values are $< \varepsilon$,
hence $t[f] \in U$. $\lrcorner$

- $U_{p_1,\dots,p_k,\frac{\varepsilon}{2}} + U_{p_1,\dots,p_k,\frac{\varepsilon}{2}} \subset U_{p_1,\dots,p_k,\varepsilon}$

 as $\|[f]+[g]\|_p \leq \|[f]\|_p + \|[g]\|_p$ for each $p \in (0,\infty)$

- $U_{p_1,\dots,p_n,\varepsilon} \cap U_{q_1,\dots,q_e,\delta} \supset U_{p_1,\dots,p_n,q_1,\dots,q_e,\min\{\varepsilon,\delta\}}$

- $\bigcap \mathcal{U} = \{0\}$

 $\ulcorner$ $[f] \neq 0 \Rightarrow \forall p : \|[f]\|_p > 0$, hence $\exists\, \varepsilon > 0$

 s.t. $\|[f]\|_p > \varepsilon \Rightarrow [f] \notin U_{p,\varepsilon}$ $\lrcorner$

$\lrcorner$

(2.) X is metrizable, as it has a ctble base of nbhds of 0.

$\ulcorner$ Let $p_1 < p_2 < \dots < p_k$ and $\varepsilon > 0$. We will show that $\exists\, \delta > 0 : U_{p_1,p_k,\delta} \subset U_{p_1,\dots,p_k,\varepsilon}$. $\Big\}$ $(*)$

To this end we will use the following inequality:

$(**)$ $\Big\{$ $p_1 < p_2 < p_3 \Rightarrow \exists\, c, d > 0$ s.t. $\int |f|^{p_2} \leq \left(\int |f|^{p_1}\right)^{c} \left(\int |f|^{p_3}\right)^{d}$

for each f measurable

It is clear that (**) implies (*) and it follows from (*) that

$$\left\{ \{ [f] ;\ \|[f]\|_{\frac{1}{n}} < \frac{1}{m} ,\ \|[f]\|_{n} < \frac{1}{m} \} ,\ m, n \in \mathbb{N} \right\}$$

is a base of nbhds of 0, so X is metrizable.

Let us prove (**):

Let $0 < p_1 < p_2 \leq p_3 < \infty$.

If $\alpha \in (0, p_2)$ and $u, v \in (1, \infty)$ satisfy $\frac{1}{u} + \frac{1}{v} = 1$,

the Hölder inequality implies

$$\int |f|^{p_2} = \int |f|^{\alpha} \cdot |f|^{p_2 - \alpha} \leq \left(\int |f|^{\alpha u} \right)^{\frac{1}{u}} \left(\int |f|^{(p_2-\alpha)v} \right)^{1/v}$$

So, if $\alpha u = p_1$, $(p_2 - \alpha) v = p_3$; we can set $c = \frac{1}{u}$, $d = \frac{1}{v}$ and we are done.

This choice is possible: $u = \frac{p_1}{\alpha}$, $v = \frac{p_3}{p_2 - \alpha}$,

α must satisfy

$$1 = \frac{1}{u} + \frac{1}{v} = \frac{\alpha}{p_1} + \frac{p_2 - \alpha}{p_3}$$

$$p_1 p_3 = \alpha p_3 + p_2 p_1 - \alpha p_1$$

$$\alpha = p_1 \cdot \frac{p_3 - p_2}{p_3 - p_1}$$

Then $\alpha \in (0, p_2)$

and if $u = \frac{p_1}{\alpha}$, $v = \frac{p_3}{p_2 - \alpha}$, it works.

(one can compute $u = \frac{p_3 - p_1}{p_3 - p_2}$, $v = \frac{p_3 - p_1}{p_2 - p_1}$)

13) X is not locally convex:

If $0<p<1$ and $\delta>0$, then $\{[f],\ \|[f]\|_p<\delta\}$ is a nbhd of 0 which contains no convex nbhd of 0

Γ If $U\subset\{[f],\ \|[f]\|_p<\delta\}$ is a convex nbhd of 0, then there is n and ε s.t.

$U_{n,\frac{1}{n},\varepsilon}\subset U$, hence $co(U_{n,\frac{1}{n},\varepsilon})\subset U\subset\{[f],\ \|[f]\|_p<\delta\}$

Fix $c>0$ s.t. $c<\varepsilon$ & $c^{1/n}<\varepsilon$

Then for each k: $c\cdot\chi_{(k,k+1)}\in U_{n,\frac{1}{n},\varepsilon}$

Γ $\|c\chi_{(k,k+1)}\|_n=c<\varepsilon$
$\|c\chi_{(k,k+1)}\|_{1/n}=c^{1/n}<\varepsilon$ $\lrcorner$

$\forall N\in\mathbb{N}$: $\frac{1}{N}\left(c\cdot\chi_{(0,1)}+c\chi_{(1,2)}+\dots+c\chi_{(N-1,N)}\right)$
$\in co\left(U_{n,\frac{1}{n},\varepsilon}\right)\subset\{[f],\ \|[f]\|_p<\delta\}$

But $\|\frac{1}{N}(c\chi_{(0,1)}+\dots+c\chi_{(N-1,N)})\|_p=$

$=\|\frac{c}{N}\chi_{(0,N)}\|_p=\left(\frac{c}{N}\right)^p\cdot N=c^pN^{1-p}\to+\infty$ for $N\to\infty$

Hence, for N large enough, this function does not belong to $\{[f],\ \|[f]\|_p<\delta\}$, a contradiction. $\lrcorner$

(5) X is an F-space

Define $\rho(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\min\{1, \|f-g\|_{1/n}\} + \min\{1, \|f-g\|_n\}\right)$

Then ρ is a translation invariant metric generating the topology of X.

Moreover, (X,ρ) is complete:

Let (f_k) be a ρ-Cauchy sequence

Then for each $n \in \mathbb{N}$ (f_k) is both $\| \|_{1/n}$-Cauchy and $\| \|_n$-Cauchy. By completeness of L^p, $p \in (0,\infty)$ for each $n \in \mathbb{N}$ there is
$g_n \in L^n(\mathbb{R})$ s.t. $f_k \xrightarrow{k} g_n$ in $L^n(\mathbb{R})$
$h_n \in L^{1/n}(\mathbb{R})$ s.t. $f_k \xrightarrow{k} h_n$ in $L^{1/n}(\mathbb{R})$

Since $f_k \to g$ in $L^p(\mathbb{R})$ $(p \in (0,\infty))$
implies $\exists\, (f_{k_n})$: $f_{k_n} \to g$ a.e.

We get $g_n = h_n$, $g_n = g_m$, $h_n = h_m$ (a.e.)

hence $\exists\, g$ s.t. $f_k \xrightarrow{k} g$ in $L^n(\mathbb{R})$ and in $L^{1/n}(\mathbb{R})$ for each $n \in \mathbb{N}$.

Then $g \in X$ and $f_k \to g$ in X.

(5)

Description of X^*

$\varphi \in X^* \Rightarrow \varphi$ is bdd on a nbhd of 0 in X

By (*) from (2) $\exists\ \varepsilon > 0\ \exists\ 0 < p < 1 < q < \infty$
s.t. φ is bdd on $U_{p,q,\varepsilon}$

Since φ is linear and $\min\{\varepsilon, \varepsilon^{1/p}\} \cdot U_{p,q,1} \subset U_{p,q,\varepsilon}$,

we see that φ is bdd on $U_{p,q,1}$.

Hence, fix $c > 0$ s.t. $\forall f \in X;\ \|f\|_p \leq 1,\ \|f\|_q \leq 1 \Rightarrow |\varphi(f)| \leq c$

Next, fix $n \in \mathbb{N}$ and set
$$X_n = \{f \in X,\ f = 0 \text{ outside } (-n,n)\}$$
Then $X_n \subset\subset X$ and for $f \in X_n$ we have

$$\|f\|_p = \int_{-n}^{n} |f|^p = \int_{-n}^{n} 1 \cdot |f|^p \leq \left(\int_{-n}^{n} 1\right)^{1-\frac{p}{q}} \left(\int_{-n}^{n} |f|^q\right)^{\frac{p}{q}} =$$

Hölder for $\frac{q}{p}$ and $\frac{q}{q-p}$

$$= (2n)^{1-\frac{p}{q}} \cdot \|f\|_q^p$$

Hence, $U_{q,(2n)^{\overset{\frac{1}{q}-\frac{1}{p}}{\boxed{\frac{1}{p}-\frac{1}{q}}}}} \cap X_n \subset U_{p,q,1} \cap X_n$,

so φ is bdd on $U_{q,(2n)^{\overset{\frac{1}{q}-\frac{1}{p}}{\boxed{\frac{1}{p}-\frac{1}{q}}}}} \cap X_n$ by c

Hence, for $f \in X_n$ we get
$$|\varphi(f)| \leq c \cdot (2n)^{\frac{1}{q}-\frac{1}{p}} \cdot \|f\|_q$$

Since $L^\infty(-n,n) \subset X_n \subset L^q(-n,n)$, we see that X_n is dense in $L^q(-n,n)$ (in the norm $\|\cdot\|_q$), so $\varphi|_{X_n}$ can be uniquely extended to an element of $(L^q(-n,n))^*$

Hence, $\exists!\ g_n \in L^r(-n,n)$ (where $\frac{1}{r}+\frac{1}{q}=1$)

s.t. $\varphi(f) = \int_{-n}^{n} f g_n$, $f \in X_n$.

By uniqueness we get $g_{n+1}\restriction(-n,n) = g_n$ (a.e.),

so there is $g : \mathbb{R} \to \mathbb{F}$ s.t.

- $g\restriction(-n,n) \in L^r(-n,n)$ for each $n \in \mathbb{N}$
- $\varphi(f) = \int_{-\infty}^{\infty} f g$, $f \in \bigcup_{n=1}^{\infty} X_n$

Now, for $f \in X$ $f = \lim_{n\to\infty} f \cdot 1_{(-n,n)}$ (limit in X)

$$\Rightarrow \varphi(f) = \lim_{n\to\infty} \varphi(f 1_{(-n,n)}) = \lim_{n\to\infty} \int_{-n}^{n} f g.$$

More precise properties of g:

Let $A = \{x \in \mathbb{R};\ |g(t)| \geq C+1\}$. Then $\lambda(A) < 1$

Indeed, suppose $\lambda(A) \geq 1$. Fix $B \subset A$ measurable with $\lambda(B)=1$

$$\text{Let } f(t) = \begin{cases} \frac{|g(t)|}{g(t)}, & t \in B \\ 0, & t \notin B \end{cases}$$

$\Rightarrow \|f\|_p = \|f\|_q = 1 \Rightarrow |\varphi(f)| \leq C$. Hence

$$C \geq |\varphi(f)| = \lim_{n\to\infty} \left|\int_{-n}^{n} f \cdot g\right| = \lim_{n\to\infty} \int_{B\cap(-n,n)} |g| = \int_B |g|$$

$\geq \lambda(B)\cdot(C+1) = C+1$, a contradiction.

Further, we claim that $\int_A |g|^r < \infty$.

⌈For any $f \in X$ we have

$$\| f \cdot \chi_A \|_p = \int_A |f|^p \underset{\substack{\uparrow \\ \text{Hölder}}}{\leq} \left(\int_A 1 \right)^{1-\frac{p}{q}} \left(\int_A |f|^q \right)^{p/q}$$

$$\leq \| f \cdot \chi_A \|_q^p$$

Hence, if we set $X_A = \{ f \in X,\ f = 0 \text{ outside } A \}$,

then $X_A \subset\subset X$ and $U_{q,1} \cap X_A \subset U_{p,q,1} \cap X_A$

So, if $f \in X_A \cap U_{q,1}$, then $|\varphi(f)| \leq C$,

hence $|\varphi(f)| \leq C \|f\|_q$, $f \in X_A$

$L^\infty(A) \subset X_A \subset L^q(A) \Rightarrow \varphi \restriction X_A$ can be uniquely

extended to an element of $(L^q(A))^*$ of norm $\leq C$.

Hence, $\exists\ h \in L^r(A)$, $\|h\|_r \leq C$ s.t.

$$\varphi(f) = \int_A f h \quad \text{for } f \in X_A$$

By uniqueness we get that for each $n \in \mathbb{N}$

$$h \restriction (-n,n) \cap A = g \restriction (-n,n) \cap A \ (\text{a.e.}), \text{ hence } h = g \restriction A \ \text{a.e.}$$

Hence, indeed $\int_A |g|^r = \|h\|_r^r \leq C^r < \infty$ ⌋

So, $g = g \cdot \chi_A + g \cdot \chi_{(\mathbb{R} \setminus A)}$, where

$g \cdot \chi_A \in L^r(\mathbb{R})$, $g \cdot \chi_{\mathbb{R} \setminus A} \in L^\infty(\mathbb{R})$

and $\varphi(f) = \int_{\mathbb{R}} f \cdot g$, $f \in X$

$$\Gamma f \in X \Rightarrow \varphi(f) = \lim_{n \to \infty} \int_{-n}^{n} fg = \lim_{n \to \infty} \left(\int_{(-n,n) \cap A} fg + \int_{(-n,n) \setminus A} fg \right)$$

$$= \underbrace{\int_A fg}_{\substack{\uparrow \\ f \in L^q,\ g\chi_A \in L^r}} + \underbrace{\int_{\mathbb{R} \setminus A} fg}_{\substack{\nwarrow \\ f \in L^1,\ g\chi_A \in L^\infty}} = \int_{\mathbb{R}} fg \quad \lrcorner$$

Conversely, if $g = g_1 + g_2$, where $g_1 \in L^r(\mathbb{R})$ for some $r \in (1, \infty)$ and $g_2 \in L^\infty(\mathbb{R})$, then

$\varphi(f) = \int_{\mathbb{R}} fg$, $f \in X$ is an element of X^*

$\Gamma f \in X \Rightarrow f \in L^1 \cap L^q$ $\left(\frac{1}{q} + \frac{1}{r} = 1\right) \Rightarrow$

$fg = fg_1 + fg_2 \in L^1 \Rightarrow \varphi$ well defined

Moreover, $U = \{f \in X;\ \|f\|_1 \le 1 \ \&\ \|f\|_q \le 1\}$ is a nbhd of 0

and $|\varphi(f)| = \left|\int_{\mathbb{R}} fg\right| \le \left|\int_{\mathbb{R}} fg_1\right| + \left|\int_{\mathbb{R}} fg_2\right| \underset{\substack{\uparrow \\ \text{Hölder}}}{\le} \|g_1\|_r + \|g_2\|_\infty$

$\Rightarrow \varphi$ is bdd on U $\Rightarrow$ $\varphi \in X^*$ $\lrcorner$

<u>Conclusion</u> $\varphi \in X^* \Leftrightarrow \exists g_1 \in L^r(\mathbb{R})$ for some $r \in (1, \infty)$ $\exists g_2 \in L^\infty(\mathbb{R})$
s.t. $\varphi(f) = \int_{\mathbb{R}} f(g_1 + g_2)$