Let X be a Frechet space, Y a LCS
$T_n : X \to Y$ continuous linear mappings
and $Tx = \lim_{n\to\infty} T_n x$ exists for each $x \in X$

Then T is a continuous linear mapping

Proof: Clearly T is linear (note that Y is a TVS,
hence $u_n \to u$ and $v_n \to v$ in Y implies $u_n + v_n \to u+v$
and $u_n \to u$ in Y implies $\lambda u_n \to \lambda u$ for $\lambda \in \mathbb{F}$)

Let us prove that T is continuous.
Fix q, any continuous seminorm on Y

Then for each $x \in X$: $q(T_n x) \to q(Tx)$
$\Rightarrow$ sequence $(q(T_n x))$ is bounded
For $m \in \mathbb{N}$ set
$M_m = \{x \in X ; \forall n \in \mathbb{N} : q(T_n x) \le m\}$
$\Rightarrow$ M_m is a closed absolutely convex set
and $X = \bigcup_m M_m$

Baire Category theorem $\Rightarrow \exists m : \operatorname{Int} M_m \neq \emptyset$
(X is metrizable by a complete metric)

$\Rightarrow \exists U$ absolutely convex open neighbourhood of 0 in X
$\exists x \in X$: $x + U \subset M_m$

M_m balanced, hence symmetric $\Rightarrow$ $-(x+U) \subset M_m$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{-x+U}$ $(U = -U)$

$\Rightarrow x + U \subset M_m$, $-x + U \subset M_m$

M_m convex $\Rightarrow U \subset M_m$
Then p_U is a continuous seminorm on X
and $q(Tx) \le m \cdot q_U(x)$
$[\, q_U(x) < c \Rightarrow q_U(\frac{x}{c}) < 1 \Rightarrow \frac{x}{c} \in U \subset M_m \Rightarrow \forall n : q(T_n \frac{x}{c}) \le m$
$\Rightarrow q(T_n x) \le m \cdot c \underset{q(T_n x) \to q(Tx)}{\Rightarrow} q(Tx) \le m \cdot c \,]$ Hence T is continuous

Let X be an F-space, Y a TVS, $T_n: X \to Y$ continuous linear mappings, $Tx = \lim_{n\to\infty} T_n x$ exists for each $x \in X$. Then T is a cts linear mapping.

Proof. • Clearly T is linear
• We will show the continuity. It is enough to show it is continuous at 0

Fix U a neighborhood of 0 in Y
Fix V – balanced neighborhood of 0 s.t. $V+V\subset U$
Fix W – balanced neighborhood of 0 s.t. $W+W\subset V$
For $m\in\mathbb{N}$ set
$$M_m = \{x\in X ;\ \forall n\in\mathbb{N}: T_n x \in m\cdot\overline{W}\}$$

$\Rightarrow M_m$ is balanced and closed
$\bigcup_m M_m = X$

$\left[x\in X \Rightarrow T_n x \to Tx \Rightarrow \{T_n x ; n\in\mathbb{N}\} \text{ is bounded in } Y \right.$
$(\{T_n x ; n\in\mathbb{N}\}\cup\{Tx\} \text{ is compact})$
$\Rightarrow \exists m: \{T_n x ; n\in\mathbb{N}\}\subset m\cdot W\subset m\cdot\overline{W}$

Baire category theorem (X is metrizable by a complete metric)
$\Rightarrow \exists m: \operatorname{Int} M_m \neq \emptyset$
$\Rightarrow \exists x\in X\ \exists G$ balanced neighborhood of 0
$x + G\subset M_m$
M_m balanced $\Rightarrow -x + G \subset M_m$

Fix $z\in G$. Then $\frac{z}{2}\in G$ as well, so $x+\frac{z}{2}, -x+\frac{z}{2} \in M_m$

$\Rightarrow \forall n: T_n(x+\frac{z}{2}), T_n(-x+\frac{z}{2}) \in m\overline{W}$,
so also $T(x+\frac{z}{2}), T(-x+\frac{z}{2}) \in m\overline{W}$
$\Rightarrow T(z) = T(x+\frac{z}{2}) + T(-x+\frac{z}{2}) \in m\overline{W} + m\overline{W}$
$\subset m(W+W) + m(W+W) \subset mV + mV$
$\subset mU$

$\Rightarrow T(G)\subset mU$, hence $T(\frac{1}{m}G)\subset U$.
This proves the continuity at 0.