

Let (X, \mathcal{T}) be a HLCS whose topology is generated by a sequence $(p_n)_{n \in \mathbb{N}}$ of seminorms

(1) WLOS $p_1 \leq p_2 \leq p_3 \leq \dots$

$\lceil q_n(x) := \max \{p_1(x), \dots, p_n(x)\}$ is also a seminorm and the family (q_n) generates the same topology as (p_n) \rfloor

(2) ~~$f(x, y)$~~ $f(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min \{1, p_n(x-y)\}$

is a translation invariant metric on X

$\lceil f(x, x) = 0$... clear

$x \neq y \Rightarrow \exists n : p_n(x-y) > 0$ (as X is Hausdorff)

Hence $f(x, y) > 0$

$f(x, y) = f(y, x)$... clear, as $p_n(x-y) = p_n(y-x)$

$f(x, z) \leq f(x, y) + f(y, z)$

\lceil for each $n \in \mathbb{N} : p_n(x-z) \leq p_n(x-y) + p_n(y-z)$

and hence also

$$\min \{1, p_n(x-z)\} \leq \min \{1, p_n(x-y)\} + \min \{1, p_n(y-z)\}$$

f translation invariant ... clear \rfloor

(3) For each $n \in \mathbb{N}$ and $\varepsilon > 0$ we have

$$\{x; p_n(x) < \varepsilon\} \subset \{x; f(x, 0) < \varepsilon + 2^{-n}\}$$

$\lceil p_n(x) < \varepsilon \Rightarrow \forall k \leq n : p_k(x) \leq p_n(x) < \varepsilon$, so

$$f(x, 0) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min \{1, p_k(x)\} = \sum_{k=1}^n \frac{1}{2^k} \min \{1, p_k(x)\} +$$

$$+ \sum_{k=n+1}^{\infty} \frac{1}{2^k} \min \{1, p_k(x)\} < \sum_{k=1}^n \frac{1}{2^k} \cdot \varepsilon + \sum_{k=n+1}^{\infty} \frac{1}{2^k} < \varepsilon + \frac{1}{2^n} \rfloor$$

$$(4) \quad \forall \varepsilon \in (0, 1) \quad \forall n \in \mathbb{N} : \left\{ x; |g(x, 0)| < \frac{\varepsilon}{2^n} \right\} \subset \left\{ x; |P_n(x)| < \varepsilon \right\}$$

$$\left[|g(x, 0)| < \frac{\varepsilon}{2^n} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{2^k} \min \{ |1, P_k(x)| \} < \frac{\varepsilon}{2^n} \right.$$

$$\Rightarrow \frac{1}{2^n} \cdot \min \{ |1, P_n(x)| \} < \frac{\varepsilon}{2^n}$$

$$\Rightarrow \min \{ |1, P_n(x)| \} < \varepsilon$$

Since $\varepsilon < 1$, it follows $P_n(x) < \varepsilon$]

(5) \mathcal{G} generates the topology \mathcal{T}

$$[(3) \Rightarrow \forall \varepsilon > 0 \quad \left\{ x; |g(x, 0)| < \varepsilon \right\} \text{ is a neighborhood of } 0$$

$$\uparrow \varepsilon > 0 \dots \exists n \in \mathbb{N} \text{ s.t. } \varepsilon + \frac{1}{2^n} < \varepsilon$$

$$\text{Then } \left\{ x; |g(x, 0)| < \varepsilon \right\} \supset \left\{ x; |g(x, 0)| < \varepsilon + \frac{1}{2^n} \right\} \supset \left\{ x; |P_n(x)| < \varepsilon \right\}$$

$$(4) \Rightarrow \left\{ x; |g(x, 0)| < \varepsilon \right\}, \varepsilon > 0, \text{ is a base of nbhd's of } 0.$$

Hence, the topology generated by \mathcal{G} has the same nbhd's of 0 as \mathcal{T} , so it coincides with \mathcal{T} .]

⑥ Let $(x_k) \subset X, x \in X$

$$f(x_k, x) \rightarrow 0 \Leftrightarrow \forall n \in \mathbb{N}: p_n(x_k - x) \rightarrow 0$$

\Rightarrow : Assume $f(x_k, x) \rightarrow 0$. Fix $n \in \mathbb{N}$

$$\varepsilon \in (0, 1) \dots \exists k_0 \forall k \geq k_0: f(x_k, x) < \frac{\varepsilon}{2^n}$$

④

$$\Rightarrow \forall k \geq k_0: p_n(x_k - x) < \varepsilon$$

\Leftarrow : Fix $\varepsilon > 0$. Find $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \frac{\varepsilon}{2}$

$$\exists k_0 \forall k \geq k_0: p_n(x_k - x) < \frac{\varepsilon}{2}$$

By ③ we have for $k \geq k_0$ $f(x_k, x) < \frac{\varepsilon}{2} + \frac{1}{2^n} < \varepsilon$

(Note: we use $f(x_k, x) = f(x_k - x, 0)$ as f is translation invariant.)

⑦ Let $(x_n) \subset X$:

(x_n) is f -Cauchy $\Leftrightarrow \forall n: (x_n)$ is p_n -Cauchy

The proof is the same as in ⑥

only we ~~use~~ work with $f(x_k, x_\ell)$ for $k, \ell \geq k_0$
or $p_n(x_k - x_\ell)$ for $k, \ell \geq k_0$