

(a) Let  $A$  be a Banach algebra with involution. Then  $A^+$  is again a  $B$ -algebra with involution if we set  $(a, \lambda)^* = (a^*, \overline{\lambda})$ ,  $(a, \lambda) \in A^+$

The only non-trivial axiom is  $((a, \lambda)(b, \mu))^* = (b, \mu)^*(a, \lambda)^*$

And this holds:

$$\begin{aligned} ((a, \lambda)(b, \mu))^* &= ((a\lambda + \lambda b + \mu a, \lambda\mu))^* = ((a\lambda + \lambda b + \mu a)^*, \overline{\lambda\mu}) = \\ &= (b^*a^* + \overline{\lambda}b^* + \overline{\mu}a^*, \overline{\lambda} \cdot \overline{\mu}) = (b^* + \overline{\mu})^*(a^*, \overline{\lambda}) = \\ &= (b, \mu)^*(a, \lambda)^*. \end{aligned}$$

(b) If  $A$  is a  $C^*$ -algebra and we define

$$\|(a, \lambda)\| = \max \{ |\lambda|; \sup_{\substack{b \in A \\ \|b\| \leq 1}} \|a\lambda + \lambda b\| \}, \quad (a, \lambda) \in A^+$$

then  $A^+$  is a  $C^*$ -algebra

① Set  $p_1(a, \lambda) := |\lambda|$ ,  $(a, \lambda) \in A^+$

Then  $p_1$  is a seminorm

$$p_1((a, \lambda)(b, \mu)) \leq p_1(a, \lambda) p_1(b, \mu) \quad [\text{in fact, } \|\cdot\| = \|\cdot\|]$$

$$p_1((a, \lambda)^*(a, \lambda)) = p_1(a, \lambda)^2$$

... this is obvious from definitions

② Set  $p_2(a, \lambda) = \sup_{\substack{b \in A \\ \|b\| \leq 1}} \|a\lambda + \lambda b\|$ ,  $(a, \lambda) \in A^+$

Interpretation: define  $L(a, \lambda)(b) = a\lambda + \lambda b$ ,  $b \in A$

Then  $L(a, \lambda) : A \rightarrow A$  linear,  $\|L(a, \lambda)\| \leq \|a\| + |\lambda|$   
 So,  $L(a, \lambda) \in L(A)$ .

Moreover,  $p_2(a, \lambda) = \|L(a, \lambda)\|$  by the definition

③  $(a, \lambda) \mapsto L(a, \lambda)$  is linear (clear)

$$L((a_1, \lambda_1)(a_2, \lambda_2)) = L(a_1, \lambda_1) L(a_2, \lambda_2)$$

$$\Uparrow L(a_1, \lambda_1) L(a_2, \lambda_2)(b) = L(a_1, \lambda_1)(a_2 b + \lambda_2 b) =$$

$$= a_1 a_2 b + \lambda_2 a_1 b + \lambda_1 a_2 b + \lambda_1 \lambda_2 b =$$

$$= L(a_1 a_2 + \lambda_2 a_1 + \lambda_1 a_2, \lambda_1 \lambda_2)(b) = L((a_1, \lambda_1)(a_2, \lambda_2))(b) \quad \Downarrow$$

④ Thus  $p_2$  is a seminorm and  $p_2((a_1, \lambda_1)(a_2, \lambda_2)) \leq p_2(a_1, \lambda_1) p_2(a_2, \lambda_2)$

$\Uparrow$  This follows from ② and ③  $\Downarrow$

⑤  $p_2((a, \lambda)^*(a, \lambda)) = p_2(a, \lambda)^2$

$\Uparrow$  It is enough to prove " $\geq$ "

$$p_2((a, \lambda)^*(a, \lambda)) = p_2((a^* a + \bar{\lambda} a + \lambda a^*, \bar{\lambda} \cdot \lambda)) =$$

$$= \sup_{\substack{b \in \mathcal{B} \\ \|b\| \leq 1}} \|a^* a b + \bar{\lambda} a b + \lambda a^* b + \bar{\lambda} \lambda b\| \geq$$

$$\geq \sup_{\|b\| \leq 1} \|b^* a^* a b + \bar{\lambda} b^* a b + \lambda b^* a^* b + \bar{\lambda} \lambda b^* b\| \geq$$

$$= \sup_{\|b\| \leq 1} \|(b^* a^* + \bar{\lambda} b^*)(a b + \lambda b)\| = \sup_{\|b\| \leq 1} \|(a b + \lambda b)^*(a b + \lambda b)\|$$

$$= \sup_{\|b\| \leq 1} \|a b + \lambda b\|^2 = p_2(a, \lambda)^2 \quad \Downarrow$$

$$(6) \quad p_2(a, 0) = \|a\| \quad \text{for } a \in A$$

$$\Uparrow p_2(a, 0) = \sup_{\substack{b \in A \\ \|b\| \leq 1}} \|ab\| \leq \sup_{\substack{b \in A \\ \|b\| \leq 1}} \|a\| \|b\| = \|a\|$$

conversely: if  $a = 0$ , then  $p_2(a, 0) = p_2(0, 0) = 0$

if  $a \neq 0$ , then

$$p_2(a, 0) \geq \left\| a - \frac{a^*}{\|a\|} \right\| = \|a\|,$$

so the equality follows.  $\Downarrow$

$$(7) \quad \|(a, \lambda)\| = \max \{ p_1(a, \lambda), p_2(a, \lambda) \}$$

$\Rightarrow \|\cdot\|$  is a seminorm satisfying

$$\|(a_1, \lambda_1) + (a_2, \lambda_2)\| \leq \|(a_1, \lambda_1)\| + \|(a_2, \lambda_2)\|$$

$$\|(a, \lambda)^* (a, \lambda)\| = \|(a, \lambda)\|^2$$

$\Uparrow$  By (1), (4), (5)  $\Downarrow$

(8)  $\|\cdot\|$  is a norm:

$$\Uparrow \|(a, \lambda)\| = 0 \Rightarrow |\lambda| = p_1(a, \lambda) = 0, \text{ i.e. } \lambda = 0$$

$$p_2(a, 0) = \|(a, 0)\| = \|a\| \quad (\text{by (6)})$$

so  $a = 0$ .  $\Downarrow$

(9)  $(A^+, \|\cdot\|)$  is a  $C^*$ -algebra

$\Uparrow$  It's enough to show that  $\|\cdot\|$  is complete

This follows, for example, from the fact that

$$\|(a_n, \lambda_n)\| \rightarrow 0 \Leftrightarrow \|a_n\| \rightarrow 0 \text{ \& } |\lambda_n| \rightarrow 0 \Leftrightarrow \|(a_n, 0)\| \rightarrow 0 \text{ \& } \|(0, \lambda_n)\| \rightarrow 0$$

$$\Rightarrow \|(a_n, \lambda_n)\| \rightarrow 0$$

Conversely:  $\|(a_n, \lambda_n)\| \rightarrow 0 \Rightarrow p_1(a_n, \lambda_n) = |\lambda_n| \rightarrow 0 \Rightarrow$

$$\Rightarrow \|(0, \lambda_n)\| \rightarrow 0 \Rightarrow \|(a_n, 0)\| = \|(a_n, \lambda_n) - (0, \lambda_n)\| \rightarrow 0$$

$\stackrel{\text{if}}{\|a_n\|}$

$\Downarrow$

(10) Suppose that  $A$  has no unit. Then  $p_2$  is a norm

$$\Gamma p_2(a, \lambda) = 0 \Rightarrow \forall b \in A \quad a b + \lambda b = 0$$

$$\lambda = 0 \Rightarrow \exists a \in A: a = 0$$

$$\text{in part. } a a^* = 0 = (0, 0)$$

$$\text{if } a \neq 0, \text{ then } \forall b \in A \Rightarrow \|a a^*\| = \|a a^*\| = 0 \Rightarrow a = 0$$

$$\lambda \neq 0 \Rightarrow \forall b \in A: b = -\frac{a}{\lambda} b \Rightarrow -\frac{1}{\lambda} b \text{ is a left unit} \Rightarrow A \text{ is unital}$$

(11) If  $A$  has no unit, then  $(A, p_2)$  is a  $C^*$ -algebra, hence  $p_2 = \|\cdot\|$ .

The equality follows from Corollary 28. So, it is enough to show that  $p_2$  is complete.

~~It is enough~~ Define  $\theta: A^+ \rightarrow \mathbb{C} \quad \theta(a, \lambda) = \lambda$ .

Then  $\theta$  is a linear functional.

$\ker \theta = \{ (a, 0) : a \in A \}$  ... it is closed as  $A$  is complete and  $p_2$  is a norm.

Then  $\theta$  is cts.

$$\text{So, } p_2(a_n, \lambda_n) \rightarrow 0 \Rightarrow \lambda_n \rightarrow 0.$$

$$p_2(0, \lambda_n) = |\lambda_n| \rightarrow 0$$

$$\text{so } \|a_n\| = p_2(a_n, 0) = p_2((a_n, \lambda_n) - (0, \lambda_n)) \rightarrow 0$$

Therefore  $p_2$  is equivalent to  $\|\cdot\|_1$ , as in (9).  $\square$

(12) If  $A$  has a unit  $e$ , then  $p_2$  is not a norm, for example  $p_2(-e, 1) = 0$ .

Then  $p_2(a, \lambda) = \|a + \lambda e\|$ , hence  $\|(a, \lambda)\| = \max\{|\lambda|, \|a + \lambda e\|\}$