Let A be a unital Banach algebra, e the unit

(i) $\rho(a)$ is an open subset of $\mathbb{C}$

⌈ Let $\lambda \in \rho(a)$. Then $\lambda e - a$ is invertible

$\mu \in \mathbb{C} \cdots \|(\mu e - a) - (\lambda e - a)\| = |\mu - \lambda|$ (*)

So, by Lemma 6 (b): $|\mu - \lambda| < \dfrac{1}{\|(\lambda e - a)^{-1}\|} \Rightarrow \mu \in \rho(a)$. ⌋

(ii) $\lambda \mapsto R(\lambda, a)$ $(= (\lambda e - a)^{-1})$ is cts on $\rho(a)$

⌈ $\lambda \mapsto \lambda e - a$ is cts $\rho(a) \to G(A)$

(it is in fact an isometry, see (*))

$x \mapsto x^{-1}$ is cts on $G(A)$ by Thm 7 (2)

Thus, $\lambda \mapsto R(\lambda, a)$ is cts, being the composition of two cts mappings. ⌋

(iii) $\lambda, \mu \in \rho(a) \Rightarrow$ $R(\mu, a) - R(\lambda, a) = -(\mu - \lambda) R(\mu, a) R(\lambda, a)$

⌈ $R(\mu, a) - R(\lambda, a) = (\mu e - a)^{-1} - (\lambda e - a)^{-1} =$

$= (\mu e - a)^{-1} (e - (\mu e - a)(\lambda e - a)^{-1}) =$

$= (\mu e - a)^{-1} ((\lambda e - a) - (\mu e - a)) (\lambda e - a)^{-1} =$

$= (\mu e - a)^{-1} (\lambda - \mu) e (\lambda e - a)^{-1} = -(\mu - \lambda) R(\mu, a) R(\lambda, a)$ ⌋

(iv) $\lambda \mapsto \varphi(R(\lambda, a))$ is holomorphic on $\rho(a)$ for each $\varphi \in A^*$

⌈ $\lambda_0 \in \rho(a)$. Then for $|\lambda - \lambda_0| < \dfrac{1}{\|(\lambda_0 e - a)^{-1}\|}$ we have $\lambda \in \rho(a)$

(by Lemma 6 (b), cf. the proof of (i) above)

and, by Lemma 6 (b) we get

$$(\lambda e - a)^{-1} = (\lambda e_0 - a + (\lambda - \lambda_0) e)^{-1} =$$

$$= (\lambda e_0 - a)^{-1} \sum_{n=0}^{\infty} (-1)^n \left((\lambda - \lambda_0) e \cdot (\lambda e_0 - a)^{-1} \right)^n =$$

$$= (\lambda e_0 - a)^{-1} \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n \left((\lambda e_0 - a)^{-1} \right)^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot (\lambda - \lambda_0)^n \left((\lambda e_0 - a)^{-1} \right)^{n+1}$$

Hence, given $\varphi \in A^*$ we have

$$\varphi(R(\lambda, a)) = \varphi((\lambda e - a)^{-1}) = \sum_{n=0}^{\infty} (-1)^n \varphi\left((\lambda e_0 - a)^{-1} \right)^{n+1} \cdot (\lambda - \lambda_0)^n$$

for $\lambda \in U\left(\lambda_0, \frac{1}{\|(\lambda_0 e - a)^{-1}\|}\right)$

Hence, $\varphi(R(\lambda, a))$ is locally a sum of a power series, hence it is a holomorphic function. $\lrcorner$

(v) $|\lambda| > \|a\| \Rightarrow \lambda \in \rho(a)$ & $R(\lambda, a) = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$

$\ulcorner$ $|\lambda| > \|a\| \Rightarrow \|\frac{a}{\lambda}\| < 1$. So $e - \frac{a}{\lambda} \in G(A) \Rightarrow \lambda e - a \in G(A)$
$\Rightarrow \lambda \in \rho(a)$, and

$$(\lambda e - a)^{-1} = (\lambda \cdot (e - \tfrac{a}{\lambda}))^{-1} = \frac{1}{\lambda} (e - \tfrac{a}{\lambda})^{-1} =$$

$$= \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{a}{\lambda}\right)^n = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}. \lrcorner$$

$\uparrow$ $\mathcal{L}6(a)$

(vi) $\lambda \in \rho(a) \Rightarrow a R(\lambda, a) = R(\lambda, a) a$

$\ulcorner$ by definition $(\lambda e - a) R(\lambda, a) = R(\lambda, a)(\lambda e - a) = e$

Hence $\cancel{\lambda R(\lambda, a)} - a R(\lambda, a) = \cancel{\lambda R(\lambda, a)} - R(\lambda, a) a$ $\lrcorner$