Invertibility in A and in A^+

(1) $a \in A \Rightarrow (a,0)$ is not invertible in A^+

Γ $(a,0)\cdot(b,\mu) = (ab+\mu a, 0) \neq (0,1)$
↑ the unit of A^+ $\rfloor$

(2) A has no unit $\Rightarrow 0 \in \sigma(a)$

Γ by definition $\sigma(a) = \sigma_{A^+}(a,0)$, by (1) $(a,0)$ is not invertible $\rfloor$

(3) A has a unit e $\Rightarrow$
(a,λ) is invertible in A^+
$\Leftrightarrow \lambda \neq 0$ & $a+\lambda e$ is invertible in A

Γ $\Rightarrow$: $(b,\mu) := (a,\lambda)^{-1}$

Then: $(a,\lambda)\cdot(b,\mu) = (0,1)$
$\parallel$
$(ab+\lambda b+\mu a, \lambda\mu)$ $\Rightarrow \lambda\mu = 1$,
so $\lambda \neq 0$
(and $\mu = \frac{1}{\lambda}$)

Moreover:
$(a+\lambda e)(b+\mu e) = \underbrace{ab+\lambda b+\mu a}_{=0} + \underbrace{\lambda\mu}_{=1} e = e$

and similarly $(b+\mu e)(a+\lambda e) = e$

$\Leftarrow$: Assume $\lambda \neq 0$ and $a+\lambda e$ is invertible in A

. We will find (b,μ) s.t. $(a,\lambda)(b,\mu) = (0,1)$

It means $ab + \lambda b + \mu a = 0$

$\lambda\mu = 1$

$\Rightarrow \mu = \frac{1}{\lambda}$, plug in the first equation:

$ab + \lambda b + \frac{1}{\lambda} a = 0$

$(a+\lambda e) b = -\frac{1}{\lambda} a$

$b := -\frac{1}{\lambda} (a+\lambda e)^{-1} a$

Hence, $\left(-\frac{1}{\lambda}(a+\lambda e)^{-1}a, \frac{1}{\lambda}\right)$ is a right inverse of (a,λ)

Similarly we find a left inverse,

hence (a,λ) is invertible $\lrcorner$

(4) A has a unit e $\Rightarrow \forall a \in A$: $\sigma_{A^+}(a,0) = \sigma_A(a) \cup \{0\}$

$\ulcorner$ by (1) $0 \in \sigma_{A^+}(a,0)$. Hence it suffices to show

that $\forall \lambda \in \mathbb{C}\setminus\{0\}$: $\lambda \in \sigma_{A^+}(a,0) \Leftrightarrow \lambda \in \sigma_A(a)$

$\lambda \in \sigma_{A^+}(a,0) \Leftrightarrow (-a,\lambda)$ is not invertible in A^+

$\overset{(3)}{\Leftrightarrow}$ $\lambda e - a$ is not invertible in A $\Leftrightarrow \lambda \in \sigma_A(a)$ $\lrcorner$