

VIII.4 Ideals, complex homomorphisms and Gelfand transform

Definition. Let A be a Banach algebra. An **ideal** in A is a proper vector subspace $I \subset A$ such that $xy \in I$ and $yx \in I$ whenever $x \in I$ and $y \in A$. A **maximal ideal** in the algebra A is an ideal, which is maximal with respect to inclusion.

Remarks:

- (1) Any ideal is a proper subalgebra. A proper subalgebra need not be an ideal.
- (2) Also **left ideals** (defined by the implication $x \in I, y \in A \Rightarrow yx \in I$) and **right ideals** (defined similarly) are studied. Then an ideal is a subspace which is both a left ideal and a right ideal. We will not investigate unilateral ideals.

Proposition 18 (properties of ideals and of maximal ideals). *Let A be a unital Banach algebra.*

- (a) *If I is an ideal in A , then $I \cap G(A) = \emptyset$.*
- (b) *The closure of an ideal in A is again an ideal in A .*
- (c) *Any ideal I in A is contained in a maximal ideal J .*
- (d) *Any maximal ideal in A is closed.*

Examples 19.

- (1) *If X is an infinite-dimensional Banach space, then $K(X)$ is a closed ideal in the Banach algebra $L(X)$.*
- (2) *The only ideal in the matrix algebra M_n (where $n \in \mathbb{N}$) is the zero ideal.*
- (3) *Let K be a compact Hausdorff space. Then all the closed ideals in the Banach algebra $\mathcal{C}(K)$ are the subspaces of the form*

$$\{f \in \mathcal{C}(K); f|_F = 0\}, \text{ where } F \subset K \text{ is a nonempty closed subset.}$$

Proposition 20 (factorization of an algebra). *Let A be a Banach algebra and let I be a closed ideal in A . Then the quotient Banach space A/I is a Banach algebra if the multiplication is defined by $q(x)q(y) = q(xy)$, where q is the quotient mapping of A onto A/I . Moreover, if A is commutative or unital, the same holds for A/I .*

Definition.

- Let A, B be Banach algebras. A mapping $h : A \rightarrow B$ is said to be a **homomorphism of Banach algebras** (shortly, a **homomorphism**), if it is linear and, moreover, $h(xy) = h(x)h(y)$ for $x, y \in A$.
- A **complex homomorphism** on a Banach algebra A is a homomorphism $h : A \rightarrow \mathbb{C}$.
- By $\Delta(A)$ we will denote the set of all the nonzero complex homomorphisms on A .

Remarks:

- (1) In the definition of a homomorphism of Banach algebras there is no continuity requirement. In some important cases a homomorphism is automatically continuous (see, e.g., Proposition 21 or Proposition 31).
- (2) If $h : A \rightarrow B$ is a homomorphism of Banach algebras, which is not identically zero, its kernel is an ideal in the algebra A .
- (3) By the preceding remark and Example 19(2) we see that for $n \geq 2$ one has $\Delta(M_n) = \emptyset$.
- (4) The quotient mapping from Proposition 20 is a homomorphism of Banach algebras.

Proposition 21 (properties of complex homomorphisms). *Let A be a Banach algebra and let $h \in \Delta(A)$.*

- If A has a unit e , then:
 - (a) $h(e) = 1$ and $\|h\| = 1$;
 - (b) $\ker h$ is a maximal ideal in A ;
 - (c) $h(x) \neq 0$ for $x \in G(A)$.
- For a general Banach algebra A (unital or not) the following hold:
 - (d) There exists a unique $\tilde{h} \in \Delta(A^+)$ extending h (i.e., such that $\tilde{h}(x, 0) = h(x)$ for $x \in A$);
 - (e) $\|h\| \leq 1$;
 - (f) $h(x) \in \sigma(x)$ for $x \in A$.

Proposition 22 (properties of $\Delta(A)$). *Let A be a Banach algebra.*

- (a) If A is unital, then $\Delta(A)$ is a weak* compact subset of the unit sphere S_{A^*} .
- (b) $\Delta(A^+) = \{\tilde{h}; h \in \Delta(A)\} \cup \{h_\infty\}$, where \tilde{h} is the extension of h provided by Proposition 21(d) and $h_\infty(x, \lambda) = \lambda$ for $(x, \lambda) \in A^+$.
- (c) If A has no unit, then $\Delta(A)$ is a subset of the unit ball B_{A^*} and $\Delta(A) \cup \{0\}$ is weak* compact. Therefore, $\Delta(A)$ is locally compact in the weak* topology.

Proposition 23 (complex homomorphisms and maximal ideals). *Let A be a unital Banach algebra.*

- (1) If I is a ideal in A of codimension one, there exists a unique $h \in \Delta(A)$ such that $I = \ker h$.
- (2) If A is commutative, then $h \mapsto \ker h$ is a bijection of $\Delta(A)$ onto the set of all the maximal ideals in A .

Definition. Let A be commutative Banach algebra.

- Let $x \in A$. For $h \in \Delta(A)$ we set $\hat{x}(h) = h(x)$. The function $\hat{x} : \Delta(A) \rightarrow \mathbb{C}$ is then called the **Gelfand transform of x** . It easily follows from definitions that \hat{x} is a continuous complex function on $\Delta(A)$, moreover by Proposition 22(c) we see that $\hat{x} \in C_0(\Delta(A))$.
- The **Gelfand transform of the algebra A** is the mapping $\Gamma : A \rightarrow C_0(\Delta(A))$ defined by $\Gamma(x) = \hat{x}$, $x \in A$.

Theorem 24 (properties of the Gelfand transform). *Let A be a commutative Banach algebra and let $\Gamma : A \rightarrow C_0(\Delta(A))$ be its Gelfand transform. Further, let $\Gamma^+ : A^+ \rightarrow C(\Delta(A^+))$ be the Gelfand transform of the algebra A^+ . To describe $\Delta(A^+)$ we use Proposition 22(b) (including the notation).*

- (a) Γ is a homomorphism of the algebra A into the algebra $C_0(\Delta(A))$.
- (b) For $(x, \lambda) \in A^+$ one has

$$\begin{aligned}\Gamma^+(x, \lambda)(\tilde{h}) &= \Gamma(x)(h) + \lambda \quad \text{for } h \in \Delta(A), \\ \Gamma^+(x, \lambda)(h_\infty) &= \lambda.\end{aligned}$$

- (c) If A is unital, then

$$\ker \Gamma = \text{rad}(A) := \bigcap \{I : I \text{ is a maximal ideal in } A\}.$$

Hence, Γ is one-to-one (and so it is an isomorphism of the algebras A and $\Gamma(A) = \hat{A}$) if and only if $\text{rad}(A) = \{0\}$ (i.e., if and only if A is **semisimple**).

- (d) Γ is one-to-one if and only if Γ^+ is one-to-one.
- (e) If A is unital, then for each $x \in A$ one has $\hat{x}(\Delta(A)) = \sigma(x)$.
- (f) If A has no unit, then for each $x \in A$ one has $\sigma(x) = \hat{x}(\Delta(A)) \cup \{0\}$.
- (g) $\|\hat{x}\| = r(x)$ for each $x \in A$.
- (h) Γ is a continuous homomorphism, one has $\|\Gamma\| \leq 1$.
- (i) Γ is a topological isomorphism of the algebras A and $\Gamma(A)$ if and only if it is one-to-one (see (c,d)) and $\hat{A} = \Gamma(A)$ is closed.
- (j) $\Gamma(A)$ separates points of $\Delta(A)$.