VII.3 Lebesgue-Bochner spaces

Definition. Let $f : \Omega \to X$ be strongly μ-measurable.

- Let $p \in [1, \infty)$. We say that the function f belongs to $L^p(\mu; X)$ (more precisely, to $L^p(\Omega, \Sigma, \mu; X)$) provided the function $\omega \mapsto \|f(\omega)\|^p$ is integrable. For such a function we set

$$\|f\|_p = \left(\int_\Omega \|f(\omega)\|^p \, \mathrm{d}\mu \right)^{1/p}.$$

- We say that f belongs to $L^\infty(\mu; X)$ (more precisely, to $L^\infty(\Omega, \Sigma, \mu; X)$) $\omega \mapsto \|f(\omega)\|$ is essentially bounded. For such a function we set

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\omega \in \Omega} \|f(\omega)\|.$$

Remarks:

(1) If $p \in [1, \infty)$, then simple integrable functions belong to $L^p(\mu; X)$. If $f = \sum_{j=1}^k x_j \chi_{E_j}$ where $E_1, \dots, E_k \in \Sigma$ are pairwise disjoint and $x_1, \dots, x_k \in X$, then

$$\|f\|_p = \left(\sum_{j=1}^k \|x_j\|^p \mu(E_j) \right)^{1/p}.$$

(2) Simple measurable functions belong $L^\infty(\mu; X)$. If f is of the above form, then

$$\|f\|_\infty = \max\{\|x_j\| ; j \in \{1, \dots, k\} \ \& \ \mu(E_j) > 0\}.$$

(3) If $p \in [1, \infty]$, $h \in L^p(\mu)$ and $x \in X$, then the function $f : \Omega \to X$ defined by the formula $f(\omega) = h(\omega) \cdot x$ belongs tp $L^p(\mu; X)$ and one has $\|f\|_p = \|h\|_p \cdot \|x\|$. We denote $f = h \cdot x$.

Theorem 14.

(a) *Let $p \in [1, \infty]$. After identifying the pairs of functions which are almost everywhere equal, the space $(L^p(\mu; X), \|\cdot\|_p)$ is a Banach space.*

(b) *The space $L^1(\mu; X)$ is formed exactly by (equivalence classes of) Bochner integrable functions.*

(c) *If X is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, the space $L^2(\mu; X)$ is a Hilbert space as well, the inner product is defined by*

$$\langle f, g \rangle = \int_\Omega \langle f(\omega), g(\omega) \rangle \, \mathrm{d}\mu(\omega), \quad f, g \in L^2(\mu; X).$$

(d) *If μ is finite, then*

$$L^\infty(\mu; X) \subset L^q(\mu; X) \subset L^p(\mu; X) \subset L^1(\mu; X).$$

whenever $1 \leq p < q \leq \infty$.

Theorem 15. *Let $p \in [1, \infty)$.*

(a) *Simple integrable functions form a dense subspace of $L^p(\mu; X)$.*

(b) *If both spaces $L^p(\mu)$ and X are separable, then $L^p(\mu; X)$ is separable as well.*

Examples 16.

(1) *Let $G \subset \mathbb{R}^n$ be a Lebesgue measurable set of strictly positive measure and let $p \in [1, \infty]$. By $L^p(G; X)$ we denote the space $L^p(\mu; X)$, where μ is the restriction of the n-dimensional Lebesgue measure to G. If $p \in [1, \infty)$ and X is separable, then $L^p(G; X)$ is separable as well.*

(2) *Let μ be the counting measure on $\mathbb{N}$ and let $p \in [1, \infty]$. Then the space $L^p(\mu; X)$ is denoted by $\ell^p(X)$ and can be represented as*

$$\ell^p(X) = \{(x_n) \in X^{\mathbb{N}}; \sum_{n=1}^{\infty} \|x_n\|^p < \infty\} \text{ pro } p \in [1, \infty),$$
$$\ell^\infty(X) = \{(x_n) \in X^{\mathbb{N}}; \sup_{n \in \mathbb{N}} \|x_n\| < \infty\}.$$

The respective norm is then defined by the formula

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p}, \quad (x_n) \in \ell^p(X), p \in [1, \infty),$$
$$\|(x_n)\|_\infty = \sup_{n \in \mathbb{N}} \|x_n\|, \quad (x_n) \in \ell^\infty(X).$$

If X is separable and $p \in [1, \infty)$, then $\ell^p(X)$ is separable as well.