V. Topological vector spaces

V.1 Linear topologies and their generating

Recalling of notation:

$\mathbb{R}$ ... the field of real numbers
$\mathbb{C}$ ... the field of complex numbers
$\mathbb{F}$ ... the field $\mathbb{R}$ or $\mathbb{C}$
If X is a vector space over $\mathbb{F}$, the zero vector is denoted by $\boldsymbol{o}$ (and sometimes by 0).
If X is a vector space over $\mathbb{F}$, $Y \subset\subset X$ means that Y is a subspace of X.

Definition. A **topological vector space over** $\mathbb{F}$ is a pair $(X, \mathcal{T})$ where X is a vector space over $\mathbb{F}$ and $\mathcal{T}$ is a topology on X with the following two properties:

(1) The mapping $(x, y) \mapsto x + y$ is a continuous mapping of $X \times X$ into X.
(2) The mapping $(t, x) \mapsto tx$ is a continuous mapping of $\mathbb{F} \times X$ into X.

The term *topological vector space* will be abbreviated by **TVS**. If $(X, \mathcal{T})$ is moreover Hausdorff, we write **HTVS**.
The symbol $\mathcal{T}(\boldsymbol{o})$ will denote the family of all the neighborhoods of $\boldsymbol{o}$ in $(X, \mathcal{T})$.

Definition. Let $(X, \mathcal{T})$ be a TVS. The space X is said to be **locally convex**, if there exists a base of neigborhoods of zero consiting of convex sets. The term *locally convex TVS* will be abbreviated by **LCS**, if it is moreover Hausdorff, then byk **HLCS**.

Examples 1.

(1) *Let $(X, \|\cdot\|)$ is a normed linear space and let $\mathcal{T}$ be the topology generated by the norm (i.e., generated by the metric induced by the norm). Then $(X, \mathcal{T})$ is HLCS.*
(2) *Let Γ be any nonempty set. Then $\mathbb{F}^\Gamma$ is HLCS, if it is equiopped by the product topology.*
(3) *The space $\mathcal{C}(\mathbb{R}, \mathbb{F})$ of continuous functions on $\mathbb{R}$ is HLCS, if it is equipped by the topology of locally uniform convergence. This topology is generated, for example, by the metric*

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, \max\{|f(x) - g(x)| \,;\, x \in [-n, n]\}\}, \quad f, g \in \mathcal{C}(\mathbb{R}, \mathbb{F}).$$

(4) *Let $\Omega \subset \mathbb{C}$ be an open set. Then the space $H(\Omega)$ of holomorphic functions on Ω is HLCS, if it is equipped by the topology of locally uniform convergence. This topology is generated, for example, by the metric*

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, \max\{|f(z) - g(z)| \,;\, z \in K_n\}\}, \quad f, g \in H(\Omega),$$

where (K_n) is an exhausting sequence of compact subsets of Ω (i.e., sequance of compact subsets satisfying $K_n \subset \operatorname{Int} K_{n+1}$ for each $n \in \mathbb{N}$ and $\bigcup_n K_n = \Omega$).
(5) *The Schwartz space $\mathscr{S}(\mathbb{R}^d)$ is HLCS, if it is equipped by the topology generated by the standard metric.*
(6) *Let $\Omega \subset \mathbb{R}^d$ be an open set and $K \subset \Omega$ a compact subset. Then the space $\mathscr{D}_K(\Omega)$ is HLCS, if it is equipped by the topology generated by the standard metric.*
(7) *Let (Ω, Σ, μ) be a measure space (where μ is a nonnegative measure) and $p \in (0, 1)$. Then the space $L^p(\Omega, \Sigma, \mu)$ consiting of equivalence classes of measurable functions $f : \Omega \to \mathbb{F}$ satisfying $\int_\Omega |f|^p \,\mathrm{d}\mu < \infty$ is HTVS, if it is equipped by the topology generated by the metric*

$$\rho(f, g) = \int_\Omega |f - g|^p \,\mathrm{d}\mu, \qquad f, g \in L^p(\Omega, \Sigma, \mu).$$

If, for example, $\Omega = [0, 1]$ and μ is the Lebesgue measure or $\Omega = \mathbb{N}$ and μ is the counting measure, then this space fails to be locally convex.

Remark: If $(X, \mathcal{T})$ is TVS, then $\mathcal{T}$ is translation invariant. I.e., if $A \subset X$ and $x \in X$, then A is open of and only if $x + A$ is open. It follows that $A \subset X$ is a neighborhood of $x \in X$ if and only if $-x + A \in \mathcal{T}(\boldsymbol{o})$. Therefore the family $\mathcal{T}(\boldsymbol{o})$ uniquely determines the topology $\mathcal{T}$.

Definition. Let X be a vector space over $\mathbb{F}$ and $A \subset X$. We say that the set A is

- **convex**, if $tx + (1-t)y \in A$ whenever $x, y \in A$ and $t \in [0,1]$;
- **symmetric**, if $A = -A$;
- **balanced**, if $\alpha A \subset A$ whenever $\alpha \in \mathbb{F}$ is such that $|\alpha| \le 1$;
- **absolutely convex**, if it is convex and balanced;
- **absorbing**, if for each $x \in X$ there exists $t > 0$ such that $\{sx; s \in [0,t]\} \subset A$.

Definition. Let X be a vector space over $\mathbb{F}$ and $A \subset X$. By the **convex hull** (**balanced hull**, **absolutely convex hull**) of A we mean the smallest convex (balanced, absolutely convex) set containing A. This set is denoted by $\mathrm{co}(A)$ ($\mathrm{b}(A)$, $\mathrm{aco}(A)$, respectively).

Proposition 2. *Let X be a vector space over $\mathbb{F}$ and $A \subset X$.*

(a) *If $\mathbb{F} = \mathbb{R}$, then A is absolutely convex, if and only if it is convex and symmetric.*
(b) $\mathrm{co}(A) = \{t_1x_1 + \cdots + t_kx_k; x_1, \dots, x_k \in A, t_1, \dots, t_k \ge 0, t_1 + \cdots + t_k = 1\}$.
(c) $\mathrm{b}(A) = \{\alpha x; x \in A, \alpha \in \mathbb{F}, |\alpha| \le 1\}$.
(d) $\mathrm{aco}(A) = \mathrm{co}(\mathrm{b}(A))$.

Proposition 3. *Let$(X, \mathcal{T})$ be a TVS and $U \in \mathcal{T}(\boldsymbol{o})$. Then:*

(i) *U is absorbing.*
(ii) *There exists $V \in \mathcal{T}(\boldsymbol{o})$ such that $V + V \subset U$.*
(iii) *There exists $V \in \mathcal{T}(\boldsymbol{o})$ open and balanced such that $V \subset U$.*
(iv) *If X is locally convex, there exists $V \in \mathcal{T}(\boldsymbol{o})$ open and absolutely convex such that $V \subset U$.*

Theorem 4.

(1) *Let $(X, \mathcal{T})$ be a TVS. Then there exists $\mathcal{U}$, a base of neighborhoods of $\boldsymbol{o}$ with the following properties:*
  (i) *The elements of $\mathcal{U}$ are absorbing, open and balanced.*
  (ii) *For any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V + V \subset U$.*
  If X is locally convex, it is possible to choose $\mathcal{U}$ such that its elements are moreover absolutely convex. If X is moreover Hausdorff, then $\bigcap \mathcal{U} = \{\boldsymbol{o}\}$.
(2) *Conversely, let X be a vector space and $\mathcal{U}$ a family of subsets of X with the following properties:*
  (i) *The elements of $\mathcal{U}$ are absorbing and balanced.*
  (ii) *For any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V + V \subset U$.*
  (iii) *For any $U, V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $W \subset U \cap V$.*
  Then there exists a unique topology $\mathcal{T}$ on X such that $(X, \mathcal{T})$ is a TVS and $\mathcal{U}$ is a base of neighborhoods of $\boldsymbol{o}$. If all the elements of $\mathcal{U}$ are moreover absolutely convex, $\mathcal{T}$ is locally convex. Further, if $\bigcap \mathcal{U} = \{\boldsymbol{o}\}$, then $\mathcal{T}$ is Hausdorff.

Examples 5.

(1) *Let T be a Hausdorff topological space and let $\mathcal{C}(T, \mathbb{F})$ denote the space of all the continuous functions on T. Then the family*

$$\mathcal{U} = \left\{ \{f \in \mathcal{C}(T, \mathbb{F}); \max_{x \in K} |f(x)| < c\}; K \subset T \text{ compact}, c > 0 \right\}$$

forms a base of neighborhood of $\boldsymbol{o}$ in a topology, in which $\mathcal{C}(T, \mathbb{F})$ is HLCS. It is the topology of uniform convergence on compact subsets. If T is locally compact, it is the topology of locally uniform convergence.
(2) *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\mathscr{D}(\Omega)$ denotes the space of all the test functions on Ω. Then the family*

$$\mathcal{U} = \{U \subset \mathscr{D}(\Omega) \text{ absolutely convex and absorbing };\ \forall K \subset \Omega \text{ compact} : U \cap \mathscr{D}_K(\Omega) \text{ is a neighborhood of zero in } \mathscr{D}_K(\Omega)\}$$

forms a base of neighborhood of $\boldsymbol{o}$ in a topology, in which $\mathscr{D}(\Omega)$ is HLCS.