PROBLEM 1

Let $\varphi : (0,1) \to \mathbb{R}$ (or to $\mathbb{C}$)

for $t \in (0,1)$ let $f(t) = \varphi \cdot \chi_{(0,t)}$,

i.e. $f(t)(u) = \varphi(u) \cdot \chi_{(0,t)}(u)$, $u \in (0,1)$.

Fix $p \in [1,\infty)$ and let $X = L^p((0,1))$.

① $f(t) \in X$ for $t \in (0,1)$

$\Longleftrightarrow$ • φ is measurable

• $\forall t \in (0,1)$ $\varphi \restriction (0,t) \in L^p((0,t))$,

i.e. $\int_0^t |\varphi|^p < \infty$

② Suppose that $f : (0,1) \to X$ (i.e. the condition from ① is satisfied).

Then f is measurable

$\ulcorner$ As X is separable, measurability = weak measurability

$X^* = L^q((0,1))$, where $\frac{1}{p} + \frac{1}{q} = 1$

$\varphi \in X^*$ .. let $g \in L^q((0,1))$ be the representing function.

Then $(\varphi \circ f)(t) = \int_0^1 f(t) \cdot g = \int_0^1 \varphi(u)\, \chi_{(0,t)}(u)\, g(u)\, du =$

$= \int_0^t \varphi \cdot g$.

Since $\varphi \cdot g \in L^1((0,t))$ for each $t \in (0,1)$, this function is cts, hence measurable

So, f is weakly measurable, hence measurable $\lrcorner$

③ Conditions for Bochner integrability:

for $t \in (0,1)$ we have

$$\|f(t)\|_X = \left(\int_0^1 |\varphi(u)\,\chi_{(0,t)}(u)|^p du\right)^{1/p} = \left(\int_0^t |\varphi|^p\right)^{1/p}$$

So, f is Bochner-integrable $\Leftrightarrow$ $\int_0^1 \left(\int_0^t |\varphi|^p\right)^{1/p} dt < \infty$

If $p=1$, then $\int_0^1 \left(\int_0^t |\varphi|\right) dt \underset{\text{FUBINI}}{=} \int_0^1 \int_u^1 |\varphi(u)|\,dt\,du = \int_0^1 (1-u)|\varphi(u)|\,du$

So, if $p=1$, then f is Bochner-integrable if and only if $\int_0^1 (1-u)|\varphi(u)|\,du < \infty$

④ Conditions for weak integrability:

f is weakly integrable $\Leftrightarrow$ $\forall \varphi \in X^*$: $\varphi \circ f$ is integrable

$\overset{②}{\Leftrightarrow}$ $\forall g \in L^q((0,1))$: $t \mapsto \int_0^t \varphi \cdot g$ is integrable

Observe this is equivalent to

$\forall g \in L^q((0,1))$: $t \mapsto \int_0^t |\varphi g|$ is integrable

[Indeed "⇑" is clear, as $\left|\int_0^t \varphi g\right| \le \int_0^t |\varphi g|$

"⇓": Let $g \in L^q(0,1)$. Set $h(t) = \begin{cases} 0 & \varphi(t) = 0 \\ \frac{|\varphi(t)\cdot g(t)|}{\varphi(t)}, & \varphi(t) \neq 0 \end{cases}$

Then $h \in L^q(0,1)$ and $h \cdot \varphi = |\varphi \cdot g|$]

So, the equivalent condition is

$$\forall g \in L^q((0,1)): \int_0^1 \left(\int_0^t |\varphi g|\right) dt < \infty$$

By Fubini theorem: $\int_0^1 \int_0^t |\varphi(u) g(u)| \, du \, dt = \int_0^1 \int_u^1 |\varphi(u) g(u)| \, dt \, du =$

$$= \int_0^1 (1-u) |\varphi(u)| \cdot |g(u)| \, du$$

So, f is weakly integrable $\Leftrightarrow$ $\forall g \in L^q((0,1))$:

$$u \mapsto (1-u)\varphi(u) g(u) \in L^1((0,1)).$$

By a consequence to Banach-Steinhaus, this is equivalent

to $u \mapsto (1-u)\varphi(u) \in L^p((0,1))$, so to $\int_0^1 (1-u)^p |\varphi(u)|^p du < \infty$

⑤ Pettis integrability:

$p \in (1,\infty) \Rightarrow X$ reflexive $\Rightarrow$ (Pettis integrability = weak integrability)

$p = 1$: The conditions for Bochner and weak integrability are the same, so Bochner i. = Pettis i. = weak i. in this case

⑥ The value of the integral:

$\varphi \in X^*$ ... $g \in L^q((0,1))$ the representing function

$$\int_0^1 \varphi \circ f = \int_0^1 \int_0^t \varphi(u) g(u) \, du \, dt \overset{(*)}{=} \int_0^1 \int_u^1 \varphi(u) g(u) \, dt \, du =$$

$$= \int_0^1 (1-u)\varphi(u) g(u) \, du = \varphi\left(u \mapsto (1-u)\varphi(u)\right)$$

In (*) we used the Fubini theorem. Its assumptions are satisfied under the weak integrability assumptions (see (4))

So, $\int_0^1 f = (u \mapsto (1-u)\varphi(u))$

(Bochner or Pettis, according to the conditions)

PROBLEM 2

Let $\varphi : (0,\infty) \to (0,\infty)$ be a function

For $t \in (0,\infty)$ let $f(t) = \mathbb{1}_{(0,\varphi(t))}$

$X = L^p(0,\infty)$, $p \in [1,\infty)$

(1) f is measurable $\Leftrightarrow$ φ is measurable

- $\forall t \in (0,\infty)$ $f(t) \in X$ (clear)

- Let $g(t) = \mathbb{1}_{(0,t)}$, $t \in (0,\infty)$

Then $\|g(t_1) - g(t_2)\|_X = |t_1 - t_2|^{1/p}$

It follows that g is a homeomorphism $(0,\infty)$ onto X

Moreover, $f = g \circ \varphi$, $\varphi = g^{-1} \circ f$

$\Rightarrow$ f is Borel-measurable $\Leftrightarrow$ φ is Borel-measurable

Since X is separable, Borel measurability = measurability

(2) $\|f(t)\| = \varphi(t)^{1/p}$

So f is Bochner-integrable iff $\int_0^\infty \varphi(t)^{1/p}\,dt < \infty$

(3) weak integrability:

f is weakly integrable $\Leftrightarrow$ $\forall g \in L^q(0,\infty)$ $\left(\frac{1}{q} + \frac{1}{p} = 1\right)$:

$t \mapsto \int_0^\infty g \cdot f(t)$ is integrable

$\qquad \parallel$

$\int_0^\infty g(u) \cdot \mathbb{1}_{(0,\varphi(t))}(u)\,du = \int_0^{\varphi(t)} g(u)\,du$

So, f is weakly integrable $\Leftrightarrow$ $\forall g \in L^q(0,\infty)$: $t \mapsto \int_0^{\varphi(t)} g$ is integrable

$\Leftrightarrow$ $\forall g \in L^q(0,\infty)$, $g \geq 0$: $t \mapsto \int_0^{\varphi(t)} g$ is integrable

$\vdash$

$\Rightarrow$ obvious

$\Leftarrow$ $g \in L^q(0,\infty) \Rightarrow |g| \in L^q(0,\infty)$ & $\left|\int_0^{\varphi(t)} g\right| \leq \int_0^{\varphi(t)} |g|$ $\dashv$

$g \in L^q$, $g \geq 0$: $\int_0^\infty \int_0^{\varphi(t)} g(u)\,du\,dt \underset{\text{FUBINI}}{=} \int_0^\infty \int_{\{t;\, u \leq \varphi(t)\}} g(u)\,dt\,du =$

$= \int_0^\infty g(u) \cdot \lambda(\varphi^{-1}([u,\infty)))\,du$

So, f weakly integrable $\Leftrightarrow$ $\forall g \in L^q(0,\infty)$, $g \geq 0$: $u \mapsto g(u) \cdot \lambda(\varphi^{-1}([u,\infty)))$ is integrable

$\Leftrightarrow$ $(u \mapsto \lambda(\varphi^{-1}([u,\infty)))) \in L^p(0,\infty)$ $\Leftrightarrow$ $\int_0^\infty \lambda(\varphi^{-1}([u,\infty)))^p\,du < \infty$

(4) Pettis integrability:

$p > 1$: Pettis integrability = weak integrability as X is reflexive

$p = 1$: $\int_0^\infty \lambda(\varphi^{-1}([u,\infty))) = \int_0^\infty \int_{\{t,\, \varphi(t) \geq u\}} 1\,dt\,du =$

$= \int_0^\infty \int_0^{\varphi(t)} 1\,du\,dt = \int_0^\infty \varphi(t)$

So, weak-integrability = Bochner integrability

Hence B-integrability = Pettis integrability

(5) The value of integral:

$\varphi \in X^*$ .. $g \in L^q(0,\infty)$ represents φ

Then $\int_0^\infty \varphi \circ f(t)\, dt = \int_0^\infty \int_0^\infty g(u) \cdot \chi_{(0,\psi(t))}(u)\, du = \int_0^\infty \int_0^{\psi(t)} g(u)\, du\, dt$

$\underset{FUBINI}{=} \int_0^\infty \int_{\psi^{-1}([u,\infty))} g(u)\, dt\, du = \int_0^\infty g(u) \cdot \lambda(\psi^{-1}([u,\infty)))\, du =$

$= \varphi\left(u \mapsto \lambda(\psi^{-1}([u,\infty))) \right)$

So, the integral is $u \mapsto \lambda(\psi^{-1}([u,\infty)))$

PROBLEM 3

Let $\psi : (0,\infty) \to \mathbb{F}$

For $t \in (0,\infty)$ let $f(t) = \psi(t) \cdot \chi_{(0,t)}$

$X := L^p((0,\infty))$, where $p \in [1,\infty)$

(1) For each $t \in (0,\infty)$ we have $f(t) \in X$

(2) measurability: X is separable, so measurability = weak measurability

$\varphi \in X^*$ ... $\exists g \in L^q(0,\infty)$ $\left(\frac{1}{q}+\frac{1}{p}=1\right)$ representing φ

$$\varphi \circ \psi(t) = \int_0^\infty g(u)\, \psi(t) \cdot \chi_{(0,t)}(u)\, du = \psi(t) \cdot \int_0^t g$$

So, f is measurable $\Leftrightarrow$ ψ is measurable

$\ulcorner$ $\Leftarrow$ $t \mapsto \int_0^t g$ is cts for each $g \in L^q$,

so $t \mapsto \psi(t) \int_0^t g$ is measurable

$\Rightarrow$ $g := \chi_{(0,1)}$ Then $\int_0^t g = \begin{cases} t, & t \le 1 \\ 1, & t \ge 1 \end{cases}$

Denote $G(t) = \int_0^t g$

by the assumption $G \cdot \psi$ is measurable, $G > 0$ and is cts

$\Rightarrow \psi = \frac{G \cdot \psi}{G}$ is measurable. $\lrcorner$

(3) Bochner integrability:

$$\|f(t)\|_X = |\psi(t)| \cdot t^{1/p}$$

So, f is Bochner integrable $\Leftrightarrow$ $\int_0^\infty |\psi(t)| \cdot t^{1/p}\, dt < \infty$

(4) weak integrability: see (2)

f is weakly integrable $\Leftrightarrow$ $\forall g \in L^q(0,\infty)$ $t \mapsto \varphi(t)\int_0^t g$ is integrable

$\Leftrightarrow \forall g \in L^q(0,\infty)$: $t \mapsto |\varphi(t)| \int_0^t |g|$ is integrable

$\Uparrow$ $\Rightarrow$ clear, $\Leftarrow$: $g \in L^q \Rightarrow |g| \in L^q$ & $\left|\int_0^t g\right| \le \int_0^t |g|$

$$\int_0^\infty |\varphi(t)| \int_0^t |g(u)|\,du\,dt \underset{\text{FUBINI}}{=} \int_0^\infty \int_u^\infty |\varphi(t)||g(u)|\,dt\,du = \int_0^\infty \left(|g(u)| \int_u^\infty |\varphi|\right) du$$

So, f is weakly integrable $\Leftrightarrow \forall g \in L^q(0,\infty)$:

$u \mapsto g(u) \int_u^\infty |\varphi|$ is integrable

$$\Leftrightarrow u \mapsto \int_u^\infty |\varphi| \in L^p(0,\infty) \Leftrightarrow \int_0^\infty \left(\int_u^\infty |\varphi|\right)^p du < \infty$$

(5) Pettis integrability:

$p > 1$ Pettis integrability = weak integrability as X is reflexive

$$p = 1: \int_0^\infty \int_u^\infty |\varphi(t)|\,dt\,du \underset{\text{FUBINI}}{=} \int_0^\infty \int_0^t |\varphi(t)|\,du\,dt = \int_0^\infty t|\varphi(t)|\,dt$$

So, Bochner integrability = weak int., hence Pettis = Bochner

(6) The value of integral:

$\varphi \in X^*$ .. represented by $g \in L^q(0,\infty)$

$$\int_0^\infty (\varphi \circ f)(t)\, dt = \int_0^\infty \int_0^\infty g(u)\, \psi(t)\, \chi_{(0,t)}(u)\, du\, dt =$$

$$= \int_0^\infty \int_0^t g(u)\, \psi(t)\, du\, dt = \int_0^\infty \int_u^\infty g(u)\, \psi(t)\, dt\, du =$$

↑ FUBINI

$$= \int_0^\infty \left(g(u) \int_u^\infty \psi(t)\, dt \right) du = \varphi \left(u \mapsto \int_u^\infty \psi \right)$$

The integral is $u \mapsto \int_u^\infty \psi$.