(a) Integrable simple functions form a vector space and $f \mapsto \int_\Omega f\,d\mu$ is linear.

$\ulcorner$ f, g simple integrable, $\alpha, \beta \in \mathbb{F}$

$$f = \sum_{j=1}^{k} x_j \chi_{E_j} \quad x_1, \dots, x_k \in X,\ E_1, \dots, E_k \in \Sigma \text{ pw disjoint},\ \bigcup E_j = \Omega$$
$$\forall j = 1 \dots k: \mu(E_j) < \infty \text{ or } x_j = 0$$
$$g = \sum_{m=1}^{\ell} y_m \chi_{A_m} \quad y_1, \dots, y_\ell \in X,\ A_1, \dots, A_\ell \in \Sigma \text{ pw disjoint},\ \bigcup A_m = \Omega$$
$$\forall m = 1 \dots \ell: \mu(A_m) < \infty \text{ or } y_m = 0$$

$$\alpha f + \beta g = \sum_{j=1}^{k} \sum_{m=1}^{\ell} (\alpha x_j + \beta y_m) \chi_{E_j \cap A_m}$$

$E_j \cap A_m \in \Sigma$ (pw disjoint, covering Ω)

If $\mu(E_j \cap A_m) = \infty$, then both $\mu(E_j) = \infty$ and $\mu(A_m) = \infty$, hence $x_j = y_m = 0 \Rightarrow \alpha x_j + \beta y_m = 0$

So, $\alpha f + \beta g$ is integrable

Moreover,
$$\int_\Omega (\alpha f + \beta g) = \sum_{j=1}^{k} \sum_{m=1}^{\ell} (\alpha x_j + \beta y_m)\, \mu(E_j \cap A_m) =$$
$$= \sum_{j=1}^{k} \sum_{m=1}^{\ell} \alpha x_j\, \mu(E_j \cap A_m) + \sum_{j=1}^{k} \sum_{m=1}^{\ell} \beta y_m\, \mu(E_j \cap A_m) =$$
$$= \alpha \sum_{j=1}^{k} x_j \left(\sum_{m=1}^{\ell} \mu(E_j \cap A_m) \right) + \beta \sum_{m=1}^{\ell} y_m \left(\sum_{j=1}^{k} \mu(E_j \cap A_m) \right) =$$
$$= \alpha \sum_{j=1}^{k} x_j\, \mu(E_j) + \beta \sum_{m=1}^{\ell} y_m\, \mu(A_m) = \alpha \int_\Omega f + \beta \int_\Omega g$$

$\ulcorner$ Any ∞ appearing in the computation is multiplied by 0 $\lrcorner$

(b) Let f be a simple measurable function
Then f is integrable $\Leftrightarrow$ $\omega \mapsto \|f(\omega)\|$ is integrable
In this case $\quad \|\int_\Omega f\,d\mu\| \leq \int_\Omega \|f\|\,d\mu$

Γ $f = \sum_{j=1}^{k} x_j \chi_{E_j}$, $E_j \in \Sigma$ pw disjoint

Then $\|f(\omega)\| = \sum_{j=1}^{k} \|x_j\| \chi_{E_j}(\omega)$, $\omega \in \Omega$

Hence it is a simple measurable function.
Moreover, since $x_j = 0 \Leftrightarrow \|x_j\| = 0$, the integrability of f and $\omega \mapsto \|f(\omega)\|$ is equivalent.

$$\int_\Omega f\,d\mu = \sum_{j=1}^{k} x_j\,\mu(E_j), \quad \int_\Omega \|f(\omega)\|\,d\mu(\omega) = \sum_{j=1}^{k} \|x_j\|\,\mu(E_j),$$

hence $\quad \|\int_\Omega f\,d\mu\| \leq \int_\Omega \|f(\omega)\|\,d\mu(\omega)$ by the triangle inequality. $\lrcorner$

(c) The limit defining the Bochner integral exists and does not depend on the choice of (f_n)

Γ Existence: Let (f_n) be a sequence of integrable simple measurable functions s.t. $\int_\Omega \|f_n(\omega) - f(\omega)\|\,d\mu(\omega) \xrightarrow{n\to\infty} 0$

$\varepsilon > 0 \Rightarrow \exists\, n_0\ \forall\, n \geq n_0 : \int_\Omega \|f_n(\omega) - f(\omega)\|\,d\mu(\omega) < \frac{\varepsilon}{2}$

Then for $m, n \geq n_0$

$$\int_\Omega \|f_m(\omega) - f_n(\omega)\|\,d\mu(\omega) \leq \int_\Omega \left(\|f_m(\omega) - f(\omega)\| + \|f(\omega) - f_n(\omega)\|\right) d\mu(\omega)$$

$$= \int_\Omega \|f_m(\omega) - f(\omega)\|\,d\mu(\omega) + \int_\Omega \|f(\omega) - f_n(\omega)\|\,d\mu(\omega) < \varepsilon$$

Thus, $\omega \mapsto \|f_m(\omega) - f_n(\omega)\|$ is an integrable simple function, so, by (b) $f_m - f_n$ is integrable and

$$\left\|\int_\Omega (f_m - f_n)\,d\mu\right\| \le \int_\Omega \|f_m(\omega) - f_n(\omega)\|\,d\mu(\omega) < \varepsilon.$$

~~By (c) we get~~ By (a) we know

$$\left\|\int_\Omega f_m\,d\mu - \int_\Omega f_n\,d\mu\right\| = \left\|\int_\Omega (f_m - f_n)\,d\mu\right\| < \varepsilon$$

So, we have proved that the sequence $\left(\int_\Omega f_n\,d\mu\right)_{n=1}^\infty$ is a Cauchy sequence in X. So, it converges.

The integral does not depend on the choice of (f_n):
If (f_n) and (g_n) are two sequences of simple integrable functions s.t. $\int \|f(\omega) - f_n(\omega)\|\,d\mu(\omega) \to 0$ and $\int \|g(\omega) - g_n(\omega)\|\,d\mu(\omega) \to 0$, then the sequence $f_1, g_1, f_2, g_2, f_3, g_3, \ldots$ satisfies the same property. $\lrcorner$

(d) Bochner integrable functions form a vector space and $f \mapsto \int_\Omega f\,d\mu$ is a linear mapping.

$\ulcorner$ f, g Bochner integrable, $\alpha, \beta \in \mathbb{F}$
(f_n) ... a sequence of simple integrable functions for f
(g_n) ——— " ——— g

then $\alpha f_n + \beta g_n$ are simple integrable functions (by (a)),

$$\int_\Omega \|\alpha f(\omega) + \beta g(\omega) - (\alpha f_n(\omega) + \beta g_n(\omega))\|\,d\mu(\omega) \le |\alpha| \int_\Omega \|f(\omega) - f_n(\omega)\|\,d\mu(\omega)$$

$$+ |\beta| \int_\Omega \|g(\omega) - g_n(\omega)\|\,d\mu(\omega) \to 0$$

So, $\alpha f+\beta g$ is B-integrable, and

$$(B)\int_\Omega (\alpha f+\beta g)\,d\mu = \lim_{n\to\infty}\int_\Omega (\alpha f_n+\beta g_n)\,d\mu \overset{(a)}{=} \lim_{n\to\infty} \alpha\int_\Omega f_n\,d\mu+\beta\int_\Omega g_n\,d\mu$$

$$= \alpha\cdot \lim_{n\to\infty}\int_\Omega f_n\,d\mu + \beta\cdot\lim_{n\to\infty}\int_\Omega g_n\,d\mu = \alpha\cdot(B)\int_\Omega f\,d\mu+\beta(B)\int_\Omega g\,d\mu$$

topology of a Banach space is linear.

$\lrcorner$

(2) f Bochner integrable $\Rightarrow$ $\omega\mapsto \|f(\omega)\|$ is integrable and

$$\|(B)\int_\Omega f\,d\mu\| \le \int_\Omega \|f(\omega)\|\,d\mu(\omega)$$

$\ulcorner$Let f be Bochner integrable. Then f is strongly μ-measurable, hence $\omega\mapsto\|f(\omega)\|$ is measurable by Prop. 1

Moreover, for ~~each~~ some $n\in\mathbb{N}$ $\int_\Omega \|f_n(\omega)-f(\omega)\|\,d\mu(\omega)<\infty$,

Since $\|f(\omega)\|\le\|f(\omega)-f_n(\omega)\|+\|f_n(\omega)\|$, we conclude that $\omega\mapsto\|f(\omega)\|$ is integrable

The estimate:

$$\|(B)\int_\Omega f\,d\mu\| = \lim_{n\to\infty}\|\int_\Omega f_n\,d\mu\| \overset{(b)}{\le} \liminf_{n\to\infty}\int_\Omega \|f_n(\omega)\|\,d\mu(\omega)$$

definition + continuity of the norm

$$\le \liminf_{n\to\infty}\int_\Omega \left(\|f_n(\omega)-f(\omega)\|+\|f(\omega)\|\right)d\mu(\omega) =$$

$$= \underbrace{\left(\liminf_{n\to\infty}\int_\Omega \|f_n(\omega)-f(\omega)\|\,d\mu(\omega)\right)}_{=0} + \int_\Omega\|f(\omega)\|\,d\mu(\omega) = \int_\Omega \|f(\omega)\|\,d\mu(\omega)$$

$\lrcorner$

(f) f Bochner integrable, $E \in \Sigma \Rightarrow \chi_E \cdot f$ is Bocher integrable

Γ (f_n) the defining sequence for f. The $\chi_E \cdot f_n$ are simple integrable functions and

$$\int_\Omega \| \chi_E(\omega) f(\omega) - \chi_E(\omega) f_n(\omega) \| \, d\mu(\omega) = \int_E \| f(\omega) - f_n(\omega) \| \, d\mu(\omega) \leq$$

$$\leq \int_\Omega \| f(\omega) - f_n(\omega) \| \, d\mu(\omega) \to 0 . \rfloor$$

Thm: f strongly measurable $\Rightarrow$ (f Bocher-integrable $\Leftrightarrow \int_\Omega \| f(\omega) \| \, d\mu(\omega) < \infty$)

Pf: $\Rightarrow$ By (e) of the previous proposition

$\Leftarrow$ Let f be strongly measurable & $\int_\Omega \| f(\omega) \| \, d\mu(\omega) < \infty$

$\exists (u_n)$, simple measurable functions s.t. $u_n \to f$ a.e.

Define

$$f_n(\omega) = \begin{cases} u_n(\omega) & \text{if } \| u_n(\omega) \| < 2 \| f(\omega) \| \\ 0 & \text{otherwise} \end{cases}$$

Then f_n is a simple function ($f_n(\Omega) \subset u_n(\Omega) \cup \{0\}$)

f_n measurable $f_n^{-1}(\{t\}) = u_n^{-1}(\{t\}) \cap \{\omega ; \| u_n(\omega) \| < 2 \| f(\omega) \| \}$

$\in \Sigma$ for $t \in u_n(\Omega)$

f_n integrable, as $\| f_n(\omega) \| \leq 2 \| f(\omega) \|$, $\omega \in \Omega$

$\Rightarrow \int \| f_n(\omega) \| \, d\mu(\omega) < \infty$ and use (b) of the previous Prop.

$f_n \to f$ a.e. Γ Fix $\omega \in \Omega$ s.t. $u_n(\omega) \to f(\omega)$

If $f(\omega) = 0$, then $f_n(\omega) = 0$ for each $n \in \mathbb{N}$

If $f(\omega) \neq 0$, then $\exists n_0 \ \forall n \geq n_0 \ \| u_n(\omega) \| < 2 \| f(\omega) \|$

so, for $n \geq n_0$ $f_n(\omega) = u_n(\omega) \to f(\omega) \rfloor$

Hence $\| f_n(\omega) - f(\omega) \| \to 0$ a.e. Moreover,

$\| f_n(\omega) - f(\omega) \| \leq \| f_n(\omega) \| + \| f(\omega) \| \leq 2 \| f(\omega) \| + \| f(\omega) \| = 3 \| f(\omega) \|$

So, by Lebesgue dom. conv. thm $\int_\Omega \| f_n(\omega) - f(\omega) \| \, d\mu(\omega) \to 0$.