(a) Simple functions, simple measurable functions, strongly $\mathcal{A}$-measurable functions, weakly $\mathcal{A}$-measurable functions form vector spaces

Γ f, g functions $M \to X$, $\alpha, \beta \in \mathbb{F}$

f, g simple $\Rightarrow \alpha f + \beta g$ is simple

Γ $(\alpha f + \beta g)(M) \subset \alpha f(M) + \beta g(M)$ $\lrcorner$

f, g simple measurable $\Rightarrow \alpha f + \beta g$ is simple measurable

Γ $f = \sum_{j=1}^{k} x_j \chi_{A_j}$ $\quad g = \sum_{\ell=1}^{m} y_\ell \chi_{B_\ell}$

$(x_1, \dots, x_k, y_1, \dots, y_m \in X$, $A_1, \dots, A_k \in \mathcal{A}$ disjoint
$B_1, \dots, B_m \in \mathcal{A}$ disjoint

Then $\alpha f + \beta g = \sum_{j=1}^{k} \sum_{\ell=1}^{m} (\alpha x_j + \beta y_\ell) \chi_{A_j \cap B_\ell}$. $\lrcorner$

f, g strongly $\mathcal{A}$-measurable $\Rightarrow \alpha f + \beta g$ is strongly $\mathcal{A}$-meas.

Γ $f = \lim u_n$, $g = \lim v_n$, u_n, v_n simple measurable
Then $\alpha f + \beta g = \lim (\alpha u_n + \beta v_n)$, $\alpha u_n + \beta v_n$ are simple measurable $\lrcorner$

f, g weakly $\mathcal{A}$-measurable $\Rightarrow \alpha f + \beta g$ is weakly $\mathcal{A}$-measurable

Γ This follows from the fact that scalar-valued measurable functions form a vector space $\lrcorner$

(b) $f_n \to f$ pointwise on M. If each f_n is Borel-$\mathcal{A}$-meas (weakly $\mathcal{A}$-meas.), then so is f

Pf For Borel $\mathcal{A}$-measurability: let $U \subset X$ be open

Then $f^{-1}(U) = \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{k=m}^{\infty} f_k^{-1}\left(\{t\in X,\ \mathrm{dist}(t, X\setminus U) > \frac{1}{n}\}\right)$

- $t \in f^{-1}(U) \Rightarrow f(t) \in U \Rightarrow \exists\, n\in\mathbb{N}$ s.t.

$U(f(t), \frac{1}{n}) \subset U \Rightarrow \exists\, m\in\mathbb{N}\ \forall k\geq m\ f_k(t) \in U(f(t), \frac{1}{n})$

So, $t \in$ the set on the RHS.

- $t \in$ RHS. Fix $n\in\mathbb{N}$ and $m\in\mathbb{N}$ s.t.

$\forall k \geq m\ \mathrm{dist}(f_k(t), X\setminus U) > \frac{1}{n}$.

Then $\mathrm{dist}(f(t), X\setminus U) \geq \frac{1}{n}$, so $f(t)\in U$, hence $t \in f^{-1}(U)$.

For weak $\mathcal{A}$-measurability

$f_n \to f \Rightarrow \forall \varphi \in X^* \ \varphi\circ f_n \to \varphi\circ f$.

So, the conclusion follows from the Borel-$\mathcal{A}$-measurability case applied to $(\varphi\circ f_n)$.

$\square$

(c) f strongly measurable $\Rightarrow f$ Borel $\mathcal{A}$-measurable $\Rightarrow f$ weakly $\mathcal{A}$-meas.
For simple functions all the types of measurability coincide

Γ • f simple. Then f is simple measurable $\Leftrightarrow f$ is Borel $\mathcal{A}$-measurable

$\Rightarrow$: $f = \sum_{i=1}^{k} x_i \chi_{A_i}$, $A_i \in \mathcal{A}$ disjoint, $x_i \in X$

$U \subset X$ open ... $f^{-1}(U) = \bigcup \{A_i ; x_i \in U\} \in \mathcal{A}$

$\Leftarrow$ f is simple $f(M) = \{x_1, \dots, x_k\}$ distinct

$A_j := f^{-1}(\{x_j\}) = M \setminus f^{-1}(X \setminus \{x_j\}) \in \mathcal{A}$.

$f = \sum_{j=1}^{k} x_j \chi_{A_j}$.

• f strongly $\mathcal{A}$-measurable $\Rightarrow f$ Borel $\mathcal{A}$-measurable

$\Gamma\Gamma$ $f = \lim u_n$, u_n simple measurable.
By the above u_n are Borel $\mathcal{A}$-meas. By (b)
f is Borel $\mathcal{A}$-measurable $\lrcorner\lrcorner$

• f Borel $\mathcal{A}$-meas. $\Rightarrow f$ weakly $\mathcal{A}$-meas.

$\Gamma\Gamma$ $\varphi \in X^*$, $U \subset \mathbb{F}$ open $\Rightarrow (\varphi \circ f)^{-1}(U) = f^{-1}(\varphi^{-1}(U)) \in \mathcal{A}$
as $\varphi^{-1}(U)$ is open in X. $\lrcorner\lrcorner$

• f simple: f weakly $\mathcal{A}$-meas $\Rightarrow f$ is simple measurable

$\Gamma\Gamma$ $f(M) = \{x_1, \dots, x_k\}$ distinct.
Fix $j \in \{1, \dots, k\}$ For $l \neq j$, $l \in \{1..k\}$ we have $x_j \neq x_l$
so $\exists \varphi_l \in X^*$ s.t. $\varphi_l(x_j) \neq \varphi_l(x_l)$
$A_j := f^{-1}(\{x_j\}) = \bigcap_{l \neq j} \{t \in M ; \varphi_l(f(t) - x_l) \neq 0\} =$
$= \bigcap_{l \neq j} (\varphi_l \circ f)^{-1}(\mathbb{F} \setminus \{x_l\}) \in \mathcal{A}$ $\lrcorner\lrcorner$ $=$ ~~$f^{-1}(\{x_l\})$~~ $\lrcorner$

(d) $f : M \to X$ strongly measurable $\Rightarrow f(M)$ is separable

$\ulcorner f = \lim u_n$, u_n simple measurable

$$\Rightarrow f(M) \subset \overline{\bigcup_{n \in \mathbb{N}} u_n(M)} \Rightarrow f(M) \text{ separable.} \lrcorner$$

(e) $f : M \to X$ Borel $\mathcal{A}$-measurable

$\Rightarrow \omega \mapsto \|f(\omega)\|$ is $\mathcal{A}$-measurable.

$\ulcorner h(x) = \|x\|$, $x \in X$ is cts on X

$\Rightarrow h \circ f$ is measurable

$(U$ open, $U \subset X \Rightarrow (h \circ f)^{-1}(U) = f^{-1}(h^{-1}(U)) \in \mathcal{A}$
as $h^{-1}(U)$ is open. $)$ $\lrcorner$