Let X be a LCS and $A \subset X$ a $\sigma(X, X^*)$-bdd set. Then A is bdd also in the original topology of X.

<u>Proof</u> (1) A is $\sigma(X,X^*)$-bdd $\Rightarrow \forall f \in X^*$: f is bdd on A.
Therefore, in case X is a normed space, this theorem is a corollary to uniform bddness principle

(2) Let $A \subset X$ be $\sigma(X, X^*)$-bdd.
To prove A is bdd, it is enough to show that any cts seminorm on X is bdd on A
So, let p be any cts seminorm on X

Set $Y := \{x \in X;\ p(x) = 0\}$
Then Y is a closed linear subspace of Y
[• Y is closed as p is cts
• $0 \in Y$ as $p(0) = 0$
• $x \in Y,\ \lambda \in \mathbb{F} \Rightarrow p(\lambda x) = |\lambda|\, p(x) = 0 \Rightarrow \lambda x \in Y$
• $x, y \in Y \Rightarrow 0 \le p(x+y) \le p(x) + p(y) = 0 \Rightarrow x + y \in Y$]

Let X/Y be the quotient in the linearly-algebraic sense and $q: X \to X/Y$ be the canonical quotient map.

Define a norm $\|\cdot\|$ on X/Y by $\|q(x)\| = p(x)$, $x \in X$
[well defined: $q(x) = q(y) \Rightarrow q(x-y) = 0 \Rightarrow x - y \in Y$,
hence $p(x-y) = 0$. So,

$$\left.\begin{aligned} p(x) &\le p(x-y) + p(y) = p(y) \\ p(y) &\le p(y-x) + p(x) = p(x) \end{aligned}\right\} \Rightarrow p(x) = p(y)$$

it is a norm: $\|0\| = \|q(0)\| = p(0) = 0$

$\|q(x)\| = 0 \Rightarrow p(x) = 0 \Rightarrow x \in Y \Rightarrow q(x) = 0$

$\|\lambda \cdot q(x)\| = \|q(\lambda x)\| = p(\lambda x) = |\lambda|\, p(x) =$
$= |\lambda| \cdot \|q(x)\|$

$\|q(x) + q(y)\| = \|q(x+y)\| = p(x+y) \le$
$\le p(x) + p(y) = \|q(x)\| + \|q(y)\|$]

q is cts $X \to (X/Y, \|\cdot\|)$ as
$\|q(x)\| \le p(x)$ (in fact "$=$") and p is
a cts seminorm on X

Further, the set $q(A)$ is weakly bdd in $(X/Y, \|\cdot\|)$
$\ulcorner$ $f \in (X/Y)^* \Rightarrow f \circ q \in X^* \Rightarrow (f \circ q)(A)$ is bdd,
and, finally, $f(q(A)) = (f \circ q)(A)$. $\lrcorner$

So, by ① (the case of normed spaces), $q(A)$ is
norm-bdd in X/Y. i.e., $\exists\, C > 0\ \forall x \in A : \|q(x)\| \le C$

But $\|q(x)\| = p(x)$. Hence, $p \le C$ on A and the proof
is finished.