Let X and Y be F-spaces and $T : X \to Y$ be a continuous linear mapping s.t. $T(X) = Y$. Then T is an open mapping

Proof: ① It is enough to prove that $\forall U \subset X$, neighborhood of 0 : TU is a nbhd of 0 in Y

② We will show : $\forall U \subset X$ nbhd of 0 : $\text{Int}\, \overline{TU} \neq \emptyset$

$\ulcorner$ U nbhd of 0 in X $\Rightarrow$ U is absorbing, hence $\bigcup_{n=1}^{\infty} nU = X$, so $\bigcup_{n=1}^{\infty} T(nU) = Y$

Y is a completely metrizable space, so by Baire category thm $\exists n \in \mathbb{N}$: $\overline{T(nU)}$ has nonempty interior.

BUT $\overline{T(nU)} = \overline{nT(U)} = n \cdot \overline{T(U)}$

(since T is linear, continuous and $y \mapsto ny$ is a homeomorphism of Y)

So, $\text{Int}\, \overline{T(nU)} = \text{Int}\, n \cdot \overline{T(U)} = n \cdot \text{Int}\, \overline{TU}$

In particular, $\text{Int}\, \overline{TU} \neq \emptyset$ $\lrcorner$

③ We will show that $\forall U \subset X$ nbhd of 0 : $\overline{TU}$ is a nbhd of 0

$\ulcorner$ $U \subset X$ nbhd of 0 $\Rightarrow$ $\exists$ V nbhd of 0 in X, $V + V \subset U$, V balanced

By ② we know that $\text{Int}\, \overline{TV} \neq \emptyset$, hence there is $y \in Y$, W a nbhd of 0 in Y s.t. $y + W \subset \overline{TV}$

$\overline{TV}$ is balanced $\Rightarrow$ $-y - W \subset \overline{TV}$ as well

Then $\overline{TU} \supset \overline{TV + TV} \supset \overline{TV} + \overline{TV} \supset (y + W) + (-y - W)$
$= W - W$, which is a nbhd of 0 in Y. $\lrcorner$

(4) We will prove: $\forall U \subset X$ nbhd of 0: TU is a nbhd of 0
Fix a complete translation invariant metric ϱ on X and set

$$U_n = \{x \in X \,;\, \varrho(x,0) < \frac{1}{2^n}\}, \quad n = 0,1,2,\dots$$

Then ~~Thus~~ (U_n) is a base of nbhds of 0 in X, it suffices to prove that TU_n is a nbhd of 0 for each n

Let us prove it for $n=0$, i.e. TU_0 is a nbhd of 0
(the general case is the same, or, replace ϱ by $2^n\varrho$):

By (3) we know that $\forall n \in \mathbb{N}$ $\overline{TU_n}$ is a nbhd of 0.
We will be done if we show $TU_0 \supset \overline{TU_1}$

To this end fix $y \in \overline{TU_1}$

Since $\overline{TU_2}$ is a nbhd of 0, ~~hence~~ we have $(y - \overline{TU_2}) \cap TU_1 \neq \emptyset$. So, there is $x_1 \in U_1$

s.t. $y - Tx_1 \in \overline{TU_2}$

Since $\overline{TU_3}$ is a nbhd of 0, we have $(y - Tx_1 - \overline{TU_3}) \cap TU_2 \neq \emptyset$

Hence there is $x_2 \in U_2$ s.t. $y - Tx_1 - Tx_2 \in \overline{TU_3}$

By induction we can find $x_n \in U_n$ for $n \in \mathbb{N}$

s.t. $y - Tx_1 - Tx_2 - \dots - Tx_n \in \overline{TU_{n+1}}$, $n \in \mathbb{N}$

Set $x_0 := \sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n$

This is well-defined, since (X,ϱ) is complete and the sum is Cauchy

Indeed, if $m > n$, then

$$\rho\left(\sum_{k=1}^{m} x_k, \sum_{k=1}^{n} x_k\right) = \rho\left(\sum_{k=n+1}^{m} x_k, 0\right) \leq$$

$$\leq \sum_{k=n+1}^{m} \rho(x_k, 0) < \sum_{k=n+1}^{m} 2^{-k} < 2^{-n},$$

where we used translation invariance of ρ.

Moreover, $x \in U_0$, since

$$\rho(x, 0) = \lim_{n\to\infty} \rho\left(\sum_{k=1}^{n} x_k, 0\right) \leq \lim_{n\to\infty} \sum_{k=1}^{n} \rho(x_k, 0) =$$

$$= \sum_{k=1}^{\infty} \rho(x_k, 0) \underset{\substack{\uparrow \\ x_k \in U_k}}{<} \sum_{k=1}^{\infty} 2^{-k} = 1$$

Finally, $Tx = y$:

$$y - Tx = \lim_{n\to\infty} (y - Tx_1 - \dots - Tx_n)$$

$$y - Tx_1 - \dots - Tx_n \in \overline{TU_{n+1}} \subset \overline{TU_k} \text{ for } n+1 > k$$

So, for each $k \in \mathbb{N}$ $y - Tx \in \overline{TU_k}$, hence $y - Tx \in \bigcap_{k=1}^{\infty} \overline{TU_k}$

To finish, observe that $\bigcap_{k=1}^{\infty} \overline{TU_k} = \{0\}$

$\ulcorner y \in Y, y \neq 0 \Rightarrow \exists V$, nbhd of 0 in Y s.t. $y \notin \overline{V}$

T cts $\Rightarrow \exists k$ s.t. $T(U_k) \subset V$

Then $\overline{T(U_k)} \subset \overline{V}$, hence $y \notin \overline{TU_k}$ $\lrcorner$