$\mathbb{F}^{\mathbb{N}}$ is a Frechet space

(1) $p_n((t_k)_{k=1}^{\infty}) := \max\{|t_1|, \dots, |t_n|\} \Rightarrow (p_n)_{n=1}^{\infty}$ is a generating family of seminorms, moreover $p_1 \le p_2 \le p_3 \le \dots$

$$\rho((t_k),(y_k)) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, p_n((t_k)-(y_k))\}$$

Then ρ is a translation invariant metric generating the topology of $\mathbb{F}^{\mathbb{N}}$ (see theorem V.22 and its proof)

(2) We will prove that ρ is complete

Let $(t^k)_{k=1}^{\infty}$ be a ρ-Cauchy sequence.

Then $\forall n$: (t^k) is Cauchy in the seminorm p_n,

i.e. $\forall \varepsilon > 0 \ \exists k_0 \ \forall k, \ell \ge k_0 : p_n(t^k - t^\ell) < \varepsilon$

WLOG $\varepsilon < 1$

$\exists k_0 \ \forall k, \ell \ge k_0 \quad \rho(t^k, t^\ell) < \frac{\varepsilon}{2^n}$

Then for $k, \ell \ge k_0$

$$\frac{1}{2^n} \min\{1, p_n(t^k - t^\ell)\} < \frac{\varepsilon}{2^n}$$
$$\Downarrow$$
$$\min\{1, p_n(t^k - t^\ell)\} < \varepsilon$$
$$\Downarrow$$
$$p_n(t^k - t^\ell) < \varepsilon$$
(as $\varepsilon < 1$) $\gg$

So, by completeness of $(\mathbb{F}^n, \|\ \|_\infty)$, the sequence $((t_1^k, \dots, t_n^k))_{k=1}^{\infty}$ converges in $\mathbb{F}^n$ to some $(y_1^n, \dots, y_n^n) \in \mathbb{F}^n$

Since the convergence in $\mathbb{F}^n$ is coordinate-wise, we get

$$y_j^n = y_j^m \quad \text{for } j \le n < m.$$

So, we get one sequence $y = (y_j)_{j=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ s.t.

$$x_j^k \xrightarrow{k\to\infty} y_j \text{ for each } j \in \mathbb{N}, \text{ i.e., } x^k \to y \text{ in } \mathbb{F}^{\mathbb{N}}.$$

$\mathcal{C}(\mathbb{R})$ is a Frechet space:

① $p_n := \max\{|f(x)|;\ x \in [-n,n]\}$, $n \in \mathbb{N}$
... it is a generating sequence of seminorms s.t. $p_1 \le p_2 \le p_3 \le \dots$

$$\rho(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, p_n(f-g)\}$$

is a translation invariant metric generating the topology of $\mathcal{C}(\mathbb{R})$

② ρ is complete:

Let $(f_k)_{k=1}^{\infty}$ be a ρ-cauchy sequence.

Then $\forall n \in \mathbb{N}$ (f_k) is cauchy in p_n

Since $p_n(f) = \| f\restriction_{[-n,n]} \|_\infty$, we get that

$(f_k\restriction_{[-n,n]})_{k=1}^{\infty}$ is uniformly cauchy on $[-n,n]$

Hence, it uniformly converges to a function

$g_n \in \mathcal{C}([-n,n])$.

So, $\forall n \in \mathbb{N}$: $\quad f_k \restriction_{[-n,n]} \rightrightarrows g_n$ on $[-n,n]$

Since the uniform convergence implies the pointwise convergence, we deduce that

$$g_n = g_m \restriction_{[-n,n]} \text{ for } m > n$$

Therefore, the function,

$$g(x) = g_n(x), \; x \in [-n,n], \; n \in \mathbb{N}$$

is a well-defined continuous function on $\mathbb{R}$ and

$f_n \overset{loc}{\rightrightarrows} g$, hence $f_n \to g$ in ρ.

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space.

1) $P_N(f) = \sup \{ (1+\|x\|^2)^N |D^\alpha f(x)| \; ; \; x \in \mathbb{R}^d, |\alpha| \leq N \}$

$\Rightarrow (P_N)_{N=0}^{\infty}$ is a generating family of (semi-)norms, moreover $P_0 \leq P_1 \leq P_2 \leq \dots$

$$\rho(f,g) = \sum_{N=0}^{\infty} \frac{1}{2^{N+1}} \min\{1, P_N(f-g)\}$$ is a translation invariant metric generating the topology of $\mathcal{S}(\mathbb{R}^d)$.

2) ρ is complete:

Let $(f_k) \subset \mathcal{S}(\mathbb{R}^d)$ be ρ-Cauchy

$\Rightarrow \forall N \in \mathbb{N}_0 \quad (f_k)$ is Cauchy in P_N

$N=0$ .. $p_0 = \| \ \|_\infty \Rightarrow (f_k)$ is uniformly cauchy,
hence $f_k \rightrightarrows f$ on $\mathbb{R}^d$ for some f, cts on $\mathbb{R}^d$ (and bdd)

Moreover, for each multiindex α and each $N \in \mathbb{N}_0$

$$\| x \mapsto (1+\|t\|^2)^N D^\alpha g(t) \|_\infty \leq p_{\max(N, |\alpha|)}(g),$$

hence $\left((1+\|t\|^2)^N D^\alpha f_k(t)\right)_{k=1}^\infty$ is uniformly cauchy on $\mathbb{R}^d$

$\Rightarrow$ there is a function $g_{\alpha,N}$ (cts and bdd on $\mathbb{R}^d$) s.t.

$$(1+\|t\|^2)^N D^\alpha f_k(t) \rightrightarrows g_{\alpha,N}(t) \quad \text{on } \mathbb{R}^d$$

Clearly, since uniform convergence implies the pointwise one,

$$g_{\alpha,N}(t) = (1+\|t\|^2)^N g_{\alpha,0}(t)$$

Moreover, by the theorem on derivatives and uniform limits,
one gets $g_{\alpha,0}(t) = D^\alpha f(t)$,

hence $g_{\alpha,N}(t) = (1+\|t\|^2)^N D^\alpha f(t)$

It follows that $f \in \mathcal{S}(\mathbb{R}^d)$

and $(1+\|t\|^2)^N D^\alpha f_k(t) \rightrightarrows (1+\|t\|^2)^N D^\alpha f(t)$ on $\mathbb{R}^d$
for each $N \in \mathbb{N}_0$ and each $\alpha \in \mathbb{N}_0^d$

Hence, $p_N(f_k - f) \to 0$ for each N, in other words,

$f_k \to f$ in $\mathcal{S}(\mathbb{R}^d)$ (hence in $\mathcal{S}$.)